Mathematical Modelling in the Natural Sciences
SS21, Exercise Sheet 3

1. Suppose there exist probabilities $s_n(t) = P(X(t) = S)$, $i_n(t) = P(X(t) = I)$ and $r_n(t) = P(X(t) = R)$ for the stochastic process $X(t) : [0, \infty) \to \{S, I, R\}$,

$$\{(s_n(t), i_n(t), r_n(t) : t \geq 0, n \in \mathbb{N}_0)\} : [0, \infty) \to \mathcal{C}^1([0, \infty))^\infty$$

solving the infinite system of differential equations,

$$\begin{cases} s_n'(t) &= \beta s_{n-1}(t) - \{\beta + n[\mu + \epsilon + \lambda \bar{I}(t)]\} s_n(t) + (n+1)[\mu + \epsilon + \lambda \bar{I}(t)] s_{n+1}(t) \\ i_n'(t) &= [(n-1)\lambda + \epsilon]\bar{S}(t) i_{n-1}(t) - [(n\lambda + \epsilon)\bar{S}(t) + n(\mu + \gamma)] i_n(t) + (n+1)(\mu + \gamma) i_{n+1}(t) \\ r_n'(t) &= \gamma \bar{I}(t) r_{n-1}(t) - [\gamma \bar{I}(t) + n\mu] r_n(t) + (n+1)\mu r_{n+1}(t) \\ & \bar{S}(t) = \sum_{n=0}^{\infty} n s_n(t), \quad \bar{I}(t) = \sum_{n=0}^{\infty} n i_n(t), \quad \bar{R}(t) = \sum_{n=0}^{\infty} n r_n(t) \end{cases}$$

for which the expected values $\bar{S}(t), \bar{I}(t), \bar{R}(t) : [0, \infty) \to \mathcal{C}^1([0, \infty))$ are well defined and satisfy

$$\bar{S}'(t) = \sum_{n=0}^{\infty} n s_n'(t), \quad \bar{I}'(t) = \sum_{n=0}^{\infty} n i_n'(t), \quad \bar{R}'(t) = \sum_{n=0}^{\infty} n r_n'(t)$$

and additionally

$$\frac{d}{dt} \sum_{n=0}^{\infty} s_n(t) = \sum_{n=0}^{\infty} s_n'(t), \quad \frac{d}{dt} \sum_{n=0}^{\infty} i_n(t) = \sum_{n=0}^{\infty} i_n'(t), \quad \frac{d}{dt} \sum_{n=0}^{\infty} r_n(t) = \sum_{n=0}^{\infty} r_n'(t).$$

Derive a 3×3 system of ODEs for $(\bar{S}, \bar{I}, \bar{R})$. Implement the model for an ever larger number of possible states and compare the expected values $\bar{S}$, $\bar{I}$ and $\bar{R}$ computed from the (stochastic) truncated infinite system with those computed from the (deterministic) 3×3 system.

2. Suppose there exists a solution $\{(s_n(t), i_n(t), r_n(t) : t \geq 0, n \in \mathbb{N}_0)\} : [0, \infty) \to \mathcal{C}^1([0, \infty))^\infty$ to the infinite system of differential equations shown above for which the expected values $\bar{S}(t), \bar{I}(t), \bar{R}(t) : [0, \infty) \to \mathcal{C}^1([0, \infty))$ are well defined and satisfy the properties stated above. Recall the endemic equilibrium state

$$S_2^\star = \frac{\mu + \gamma}{\lambda}, \quad I_2^\star = \frac{\beta}{\mu + \gamma} - \frac{\mu}{\lambda}, \quad R_2^\star = \frac{\gamma}{\mu} \left[\frac{\beta}{\mu + \gamma} - \frac{\mu}{\lambda} \right]$$

and suppose further with $I_2^\star > 0$ that the following hold:

$$i_n(0) \in (0, 1), \quad \forall n \in \mathbb{N}_0, \quad \sum_{n=0}^{\infty} i_n(0) = 1 \quad \text{and} \quad (\bar{S}(0), \bar{I}(0), \bar{R}(0)) = (S_2^*, I_2^*, R_2^*).$$

Show that $i_n(t) \to \delta_{n,0}$, $t \to \infty$. Implement the truncated model for an ever larger number of possible states starting always with an initial condition corresponding to the endemic equilibrium.

3. For $N = 2$, $M = 1$ and given $\beta, \gamma, \lambda, \mu$ (and possibly ϵ) of the cellular automaton on pages 137 – 142 of the lecture notes, write the transition probabilities of the cellular automaton as a stochastic matrix and find the equilibrium. Develop a general method for larger N and M so that results can be compared with the Monte-Carlo simulations from the lecture.

4. Implement the Monte-Carle simulation of the cellular automaton on pages 137 – 142 of the lecture notes.

5. Develop a continous time lumped parameter infection model which possesses periodic solutions and perhaps even period doubling transitioning to chaos. Feel free to include any relevant effects, even such as loss of immunity, vaccination, etc. This challenge was met by Elias Windisch who contributed page 126 in the script!

6. Develop a discrete time lumped parameter infection model which possesses periodic solutions and perhaps even period doubling transitioning to chaos. Feel free to include any relevant effects, even such as loss of immunity, vaccination, etc.