Mathematical Modelling in the Natural Sciences
SS21, Exercises, Sheet 2

1. Show that the equilibrium $(x^\star, y^\star) = (a_2/b_2, a_1/b_1)$ for the Gause equations,

$$x'(t) = (a_1 - b_1 y)x, \quad y'(t) = (a_2 - b_2 x)y, \quad a_1, a_2, b_1, b_2 > 0$$

is unstable and, in particular, it corresponds to a saddle point.

2. Let the damped mass spring model,

$$mu''(t) = f - ku(t) - cu'(t)$$

for $m = 1$, $k = 1$, $f = 1$ and $c = 3$ be written in first order form

$$\boldsymbol{u}'(t) = A\boldsymbol{u}(t) + \boldsymbol{b}, \quad \boldsymbol{u}(t) = (u(t); u'(t)), \quad A = [0, 1; -k, -c]/m, \quad \boldsymbol{b} = (0; f/m).$$

First show that the equilibrium

$$\boldsymbol{u}^\star = (u^\star; 0) = -A^{-1}\boldsymbol{b}$$

is locally asymptotically stable. Then note that the system can also be written as

$$\boldsymbol{w}'(t) = A\boldsymbol{w}(t), \quad \boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^\star.$$

To show that $\boldsymbol{w} = \boldsymbol{0}$ is globally asymptotically stable, derive a function $P(\boldsymbol{w})$ satisfying

$$S^{-\top} S^{-1} A\boldsymbol{w} = -\nabla P(\boldsymbol{w}), \quad AS = S\Lambda, \quad \Lambda = \text{diag}\{\lambda_i\}_{i=1}^2$$

and show that P decreases to its global minimum at $\boldsymbol{w} = \boldsymbol{0}$ along every solution $\boldsymbol{w}(t)$.

3. Show that the function

$$F(x, y) = -a_2 \ln(x) + b_2 x - a_1 \ln(y) + b_1 y$$

is a Lyapunov function for the predator-prey model,

$$x' = (a_1 - b_1 y)x, \quad y' = (b_2 x - a_2)y, \quad a_1, a_2, b_1, b_2 > 0.$$

4. Let $p_n(t) = P\{X(t) = n\}$ be the probability that n customers are waiting to be served at time t, and there are two cashiers. The system of ODEs for these probabilities is:

$$\begin{aligned} p_0'(t) &= -b_0 p_0(t) + d_1 p_1(t) \\ p_n'(t) &= b_{n-1} p_{n-1}(t) - (b_n + d_n) p_n(t) + d_{n+1} p_{n+1}(t), \quad 1 \le n \le N-1 \\ p_N'(t) &= b_{N-1} p_{N-1}(t) - d_N p_N(t) \end{aligned}$$

where the coefficients $\{b_n\}$ and $\{d_n\}$ are determined as follows.

- The average time between customer arrivals is $c = 1/b_n$ and is independent of the number of cashiers.
- If there is only one customer, then $s = 1/d_1$ is the average service time when only one cashier is in operation.

- When there are at least two customers, then $s/2 = 1/d_n$, $2 \le n \le N$, is the average service time when two cashiers are in operation.

This information is summarized as follows:

$$b_n = 1/c, \quad d_n = \begin{cases} 2/s, & 2 \le n \le N \\ 1/s, & n = 1. \end{cases}$$

Let $\{p_n^\star\}$ be the stationary state for $X(t)$. Show with $\rho = s/c$,

$$E[X^\star] = p_0 \rho \frac{N(\rho/2)^{N+1} - (N+1)(\rho/2)^N + 1}{(1-\rho/2)^2}, \quad p_0 = \frac{1-\rho/2}{1+\rho/2 - \rho(\rho/2)^N}$$

and

$$E[X^\star] \overset{\rho \to 2}{\longrightarrow} \frac{N(N+1)}{1+2N}, \quad E[X^\star] \overset{N\to\infty}{\longrightarrow} \frac{4\rho}{4-\rho^2} \equiv L_2(\rho) \quad \text{für } \rho \in (0,2).$$

5. Let $i \in \{0(\mathrm{G}), 1, \dots, N\}$ be an index for the floor of a building. Let X be a random variable which satisfies $X(n) = i$ if an elevator is at the ith floor after n time steps each of duration Δt. For $p_i(n) = P(X(n) = i)$ and $\boldsymbol{p}(n) = \{p_i(n)\}_{i=0}^N$ let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be a stochastic matrix, where $\boldsymbol{p}(n) = P^\top \boldsymbol{p}(n-1)$ holds.

   It is assumed that Δt is so small that jumps of two or more floors in a time interval of length Δt are not possible. Otherwise all transitions with neighboring floors is possible, also that there be no transition. Thus, P is genuinely tridiagonal but otherwise an arbitrary stochastic matrix. In particular, it is not necessarily the case that P is symmetric.

   For an $N \in \mathbb{N}$ choose such a stochastic matrix P and carry out the following calculations. Find a stationary state (equilibrium) of the states $\boldsymbol{p}^\star$ of the elevator. Confirm that $\boldsymbol{p}_0^\top P^n \to \boldsymbol{p}^{\star\top}$, $n \to \infty$, holds for an arbitrary initial distribution $\boldsymbol{p}_0$. Confirm further that $P^n > 0$ holds for an $n \in \mathbb{N}$, and that all rows of P^n converge to the state $\boldsymbol{p}^{\star\top}$ for ever increasing n. How can the theorem from the lecture about such chains be applied here? Under which conditions are all the entries of $\boldsymbol{p}^\star$ equal?

   With the same P and $\boldsymbol{p}^\star$ make the following random walk. Choose an arbitrary floor initially with index i. Determine a next floor j according to the transition probability $P_{i,j}$. This can be done, e.g., with a uniformly distributed random variable z in $[0,1]$: $j = i-1$ if $z \in [0, P_{i,i-1}]$, $j = i$ if $z \in (P_{i,i-1}, P_{i,i-1} + P_{i,1}]$ and $j = i+1$ if $z \in (P_{i,i-1} + P_{i,1}, 1]$. Then overwrite i with j and carry out such steps several times until the relative frequency distribution for the floors becomes stable. Compare this relative frequency distribution with $\boldsymbol{p}^*$.

6. For the parameters $\beta = 100$, $\mu = 0.001$, $\gamma = 0.4$ and $\lambda = 5 \cdot 10^{-6}$ implement the SIR model,

$$S' = \beta - (\mu + \lambda I)S, \quad I' = (\lambda S - \mu - \gamma)I, \quad R' = -\mu R + \gamma I$$

   and plot the results in time and in phase space. Show that the equilibrium obtained is locally asymptotically stable.