

Mathematical Modelling in the Natural Sciences

SS21, Exercises, Sheet 1

1. For the *Supersize Me!* problem suppose that the following data have been measured:

```
n = 30;
t = linspace(0,30,n);
p = 1.0;
m = 231.48 + (84-231.48)exp(-t/361.11) + p*randn(1,n);
```

Formulate a method for estimating the parameter ϕ in the model

$$m'(t) = \epsilon/\kappa - m(t)\phi/\kappa, \quad m(0) = 84, \quad \epsilon = 5000, \quad \kappa = 7800$$

and implement this method using Matlab. For different values of the noise level p , compare your result with the value of $\phi = 21.6$ used in the lecture notes. Hint: Consider using the ℓ_2 - or the ℓ_1 -norm of the differences between simulated and measured data.

2. Develop a refinement of the *Supersize Me!* model which is intended to be more suitable for (a) very small or (b) very large values of m , and argue why your model is more suitable for these extreme cases. Hint: Consider the behavior of the steady state with respect to other parameters.
3. For the system of differential equations modelling the discovery of treasures by a random walk

$$\begin{aligned} p'_0(t) &= -\beta p_0(t) \\ p'_n(t) &= -\beta p_n(t) + \beta p_{n-1}(t), \quad n = 1, 2, \dots, N-1 \\ p'_N(t) &= \beta p_{N-1}(t) \end{aligned} \quad 0 \leq t \leq T$$

determine the initial conditions which correspond to not having discovered any treasures at all at time $t = 0$. Choose values $\beta, T > 0$ and $N \in \mathbb{N}$ and solve the above system using Matlab plotting $\{p_n(t)\}_{n=0}^N$ together for $t \in [0, T]$. Given the values $\{p_n(t) : t \in [0, T]\}_{n=0}^N$ determine $E_N(t)$ the expected number of treasures discovered up to time t . Plot also $E_N(t)$ together with $E(t)$ presented in the lecture notes, and compare the two plots for ever larger N .

4. Let the data be given:

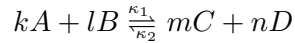
$$Y = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

- (a) Carry out PCA to determine Y_c , $K = \frac{1}{4}Y_c Y_c^\top$, V and $\Lambda = \text{diag}\{\lambda_i\}$ with $KV = V\Lambda$ and $Y_s = \Lambda^{-\frac{1}{2}}V^\top Y_c$. Plot Y and Y_s .
- (b) Show that the sphered data satisfy $\frac{1}{4}Y_s Y_s^\top = I$. Show that when the sphered data are projected onto an arbitrary axis $\hat{\mathbf{w}} \in \mathbb{R}^2$ with $\|\hat{\mathbf{w}}\|_{\ell_2} = 1$, the projected data have the variance 1.
- (c) For $\theta \in [0, 2\pi]$ define $\mathbf{u} = (\cos(\theta), \sin(\theta))$, $\mathbf{u}^\perp = (-\sin(\theta), \cos(\theta))$ and $U(\theta) = (\mathbf{u}(\theta); \mathbf{u}^\perp(\theta))$. Plot $J(\theta) = \mathcal{K}^2(\mathbf{u}(\theta)Y_s)$ where \mathcal{K} is the kurtosis. Using this plot, determine the value θ^* which maximizes J . Set $X_c = U(\theta^*)Y_s$. Assume that the columns of Y have equal probability, and show that the coordinates (x, y) of columns of the resulting X_c are statistically independent, i.e.,

$$P(x = \alpha \ \& \ y = \beta) = P(x = \alpha) \cdot P(y = \beta).$$

Bonus: Apply PCA to compress these [data](#) down to 10 principal components and thereby denoise the image sequence. Hint: This [article](#) may be helpful.

5. With dimensional analysis derive the third Kepler Law: *The squares of the orbital periods of two planets are proportional to the cubes of their semi-major axes.* (Hint: See page 35 in the [script](#), and don't forget to verify the conditions of the Buckingham Pi Theorem.)
6. The chemical reaction



is given with the following parameters:

$$k = 2, \quad l = 1, \quad m = 2, \quad n = 1$$

$$[A](0) = 2, \quad [B](0) = 2, \quad [C](0) = 2, \quad [D](0) = 1.$$

Determine the initial value problem

$$x'(t) = f(x(t); \kappa_1, \kappa_2), \quad x(0) = x_0$$

where

$$\begin{aligned} [A](t) &= [A](0) - kx(t), & [B](t) &= [B](0) - lx(t) \\ [C](t) &= [C](0) + mx(t), & [D](t) &= [D](0) + nx(t). \end{aligned}$$

Rewrite the equation $f(x^*; \kappa_1, \kappa_2) = 0$ in the form,

$$\kappa_2/\kappa_1 = r(x^*)$$

giving a relationship between equilibria and the quotient κ_2/κ_1 . (Note that the condition $\kappa_2/\kappa_1 > 0$ implies, through the form of r , that an equilibrium must satisfy $x^* > -1$.) Show that any equilibrium x^* with $r'(x^*) < 0$ is (locally asymptotically) stable, while any equilibrium x^* with $r'(x^*) > 0$ is unstable. (Hint: It holds that $f'(x^*; \kappa_1, \kappa_2)/\kappa_1 = 4r'(x^*)(1 + x^*)^3$.) Derive a corresponding potential landscape $p(x, \kappa_2/\kappa_1)$, where $f(x; \kappa_1, \kappa_2)/\kappa_1 = -p_x(x, \kappa_2/\kappa_1)$. For various values of κ_2/κ_1 , plot $f(x; \kappa_1, \kappa_2)$ and $p(x, \kappa_2/\kappa_1)$ in the interval $0 \leq x \leq 3$ to demonstrate the associated hysteresis graphically.