Mathematical Modelling in the Natural Sciences SS20
Solutions to Exercises on Sheet 1

Exercises und Lecture Notes

Contents

• Exercise 1: *Supersize Me*

◦ Task

For the *Supersize Me!* problem suppose that the following data have been measured:

```
n = 30;
t = linspace(0,30,n);
p = 1.0;
m = 231.48 + (84-231.48)exp(-t/361.11) + p*randn(1,n);
```

Formulate a method for estimating the parameter ϕ in the model

$$m'(t) = \epsilon/\kappa - m(t)\phi/\kappa, \quad m(0) = 84, \quad \epsilon = 5000, \quad \kappa = 7800$$

and implement this method using Matlab. For different values of the noise level p, compare your result with the value of $\phi = 21.6$ used in the lecture notes. Hint: Consider using the ℓ_2- or the ℓ_1-norm of the differences between simulated and measured data.

◦ Solution

A Matlab function for the solution to this initial value problem for a given value of `p` $(= \phi)$ is given as follows.

```
function m = super(p)

% for a given p in the ODE,
%     m' = (e/k) - (p/k)*m,  m(0) = m0
% compute the time course of the mass m over 30 days

% energy flow per day
    e = 5000;
% density kcal/kg
    k = 7800;

% ODE to solve is m' = dm(t,m)
    dm = @(t,m) (e/k) - (p/k)*m;
```

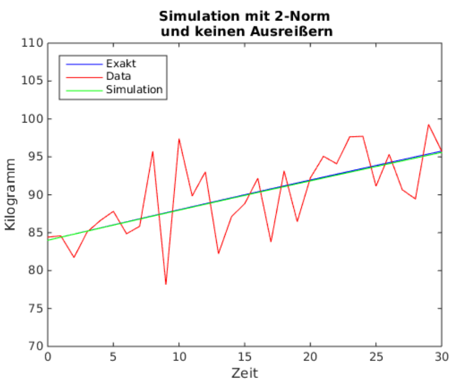
```
% initial mass
    m0 = 84;
% time points in which the solution should be output
    tpte = (0:30)';
% options
    opts = odeset('RelTol',1.0e-6);
% solution of the ODE
    [t,m] = ode45(dm,tpte,m0,opts);

end
```

This function is called from the following Matlab code to estimate a value of `p` ($= \phi$) which fits the error level `ns`.

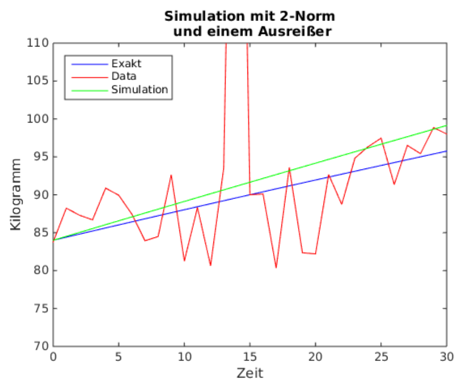
```
function S01Exp1

% determine the value of p in the ODE
%     m' = (e/k) - (p/k)*m,  m(0) = m0
% where the difference between the computed solution and simulated data
% is minimal in a chosen sense.

% energy outflow parameter, the target exact value
    pexakt = 21.6;
% energy flow per day
    e = 5000;
% density, kcal/kg
    k = 7800;
% initial mass
    m0 = 84;
% time points in which the solution should be output
    tpte = (0:30)';
% exact solution of the ODE m' = (e/k) - (p/k)*m,  m(0) = m0
    mexakt = (e/pexakt) + (m0 - e/pexakt)*exp(-tpte*pexakt/k);
% noise in the data
    ns = 5;
% noisy solution, simulated data for the determination of p
    mdata  = mexakt + (ns*m0/100)*randn(31,1); %  mdata(15)=mdata(15)*2;

% objective function to minimize for the detemrination of p,
    f = @(p) norm(mdata - super(p),1);
% use 1-norm (here) or say the 2-norm

% initial value for the estimate of p
    p0 = 20;
% options for minimization
    opts = optimset('MaxFunEvals',1000,'TolX',1.0e-6);
% minimization for the determination of p
    p = fminsearch(f,p0,opts);
```

```
% results
    disp(['Exakter Wert = ',num2str(pexakt)])
    disp(['Rauschengrad in Daten = ',num2str(ns),'%'])
    disp(['Abgeschaetzter Wert = ',num2str(p)])

% grafical representation
    plot(tpte,mexakt,'b',tpte,mdata,'r',tpte,super(p),'g')
    axis([0 30 70 110])
    xlabel('Zeit')
    ylabel('Kilogramm')
    legend('Exakt','Data','Simulation','Location','northwest')
    title(sprintf('Simulation mit 2-Norm\n und keinen Ausreiern'))
%   title(sprintf('Simulation mit 2-Norm\n und einem Ausreier'))
%   title(sprintf('Simulation mit 1-Norm\n und einem Ausreier'))

end
```

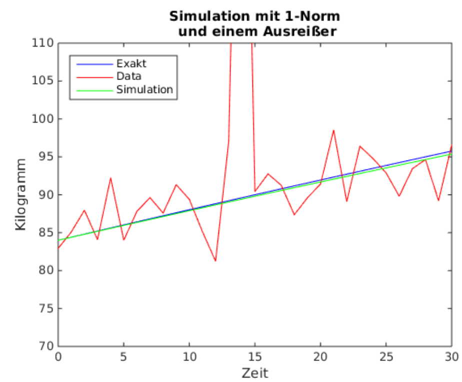
Results with ns=5% noise and with (a) ℓ_2-norm and no outliers, (b) ℓ_2-norm and one outlier, (c) ℓ_1-norm and one outlier are shown here with the estimated value of $\phi^\star$:



(a) $\phi^\star = 22.1$



(b) $\phi^\star = 11.6$



(c) $\phi^\star = 22.8$

• Exercise 2: *Supersize Me* Supplement

◦ Task

Develop a refinement of the *Supersize Me!* model which is intended to be more suitable for (a) very small or (b) very large values of m, and argue why your model is more suitable for these extreme cases. Hint: Consider the behavior of the steady state with respect to other parameters.

◦ Solution

For the *Supersize Me!* model in the lecture notes the equilibrium satisfies

$$m^\star(\epsilon) = \epsilon/\phi$$

and therefore increases linearly with respect to ϵ. This is not realistic, since the efficiency of energy storage in the body presumably decreases as more energy is consumed. The next level of complexity for the relationship between the equilibrium $m^\star$ and the energy inflow ϵ is

$$m^\star(\epsilon) = \sqrt{\epsilon/\phi}$$

which is consistent with the new model,

$$m'(t) = \epsilon/\kappa - m^2(t)\phi/\kappa, \quad m(0) = m_0 = 84, \quad \epsilon = 5000, \quad \kappa = 7800.$$

The solution of thsi initial value problem is

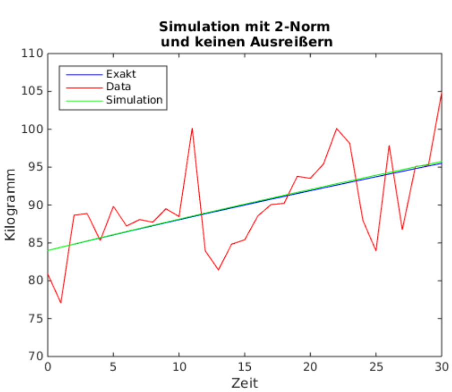
$$m(t) = \sqrt{\frac{\epsilon}{\phi}} \tanh\left[\frac{\sqrt{\epsilon\phi}}{\kappa} t + \tanh^{-1}\left(m_0\sqrt{\frac{\phi}{\epsilon}}\right)\right]$$

The Matlab code from the last example can be modified as follows for the new model.

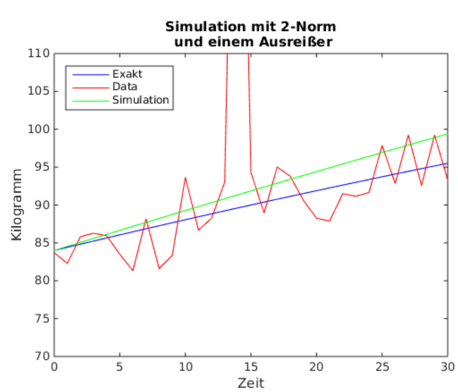
```
% for a given p in the ODE,
%     m' = (e/k) - (p/k)*m^2,  m(0) = m0

...

% ODE to solve is m' = dm(t,m)
    dm = @(t,m) (e/k) - (p/k)*m^2;

...

% energy outflow parameter, the target exact value
    pexakt = 0.248;

...

% exact solution of the ODE m' = (e/k) - (p/k)*m,  m(0) = m0
    mexakt = sqrt(e/pexakt)*tanh(sqrt(e*pexakt)*tpte/k ...
                                    +atanh(m0*sqrt(pexakt/e)));
```

where the exact value $\phi = 0.248$ corresponds to the final mass $m(30) = 95.5$. (Note that the value $\phi = 21.6$ for the the rule of thumb in the previous example is determined empirically in the same way.)

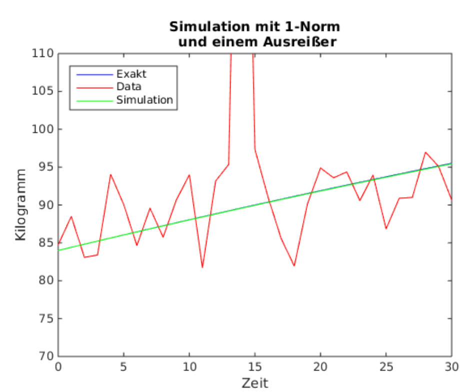
Results with `ns=5%` noise and with (a) ℓ_2-norm and no outliers, (b) ℓ_2-norm and one outlier, (c) ℓ_1-norm and one outlier are shown here with the estimated value of $\phi^\star$:



(a) $\phi^\star = 0.239$



(b) $\phi^\star = 0.118$



(c) $\phi^\star = 0.250$

• Exercise 3: Stochastic Discovery

◦ Task

For the system of differential equations modelling the discovery of treasures by a random walk

$$\begin{aligned} p_0'(t) &= -\beta p_0(t) \\ p_n'(t) &= -\beta p_n(t) + \beta p_{n-1}(t), \quad n = 1, 2, \ldots, N-1 \qquad 0 \le t \le T \\ p_N'(t) &= \beta p_{N-1}(t) \end{aligned}$$

determine the initial conditions which correspond to not having discovered any treasures at all at time $t = 0$. Choose values $\beta, T > 0$ and $N \in \mathbb{N}$ and solve the above system using Matlab plotting $\{p_n(t)\}_{n=0}^N$ together for $t \in [0, T]$. Given the values $\{p_n(t) : t \in [0, T]\}_{n=0}^N$ determine $E_N(t)$ the expected number of treasures discovered up to time t. Plot also $E_N(t)$ together with $E(t)$ presented in the lecture notes, and compare the two plots for ever larger N.

◦ Solution

That no treasures have been discovered at the time $t = 0$ corresponds to the initial state,

$$p_0(0) = 1, \quad p_1(0) = 0, \quad \ldots \quad p_N(0) = 0, \quad \text{d.h. } p_i(0) = \delta_{i,0}.$$

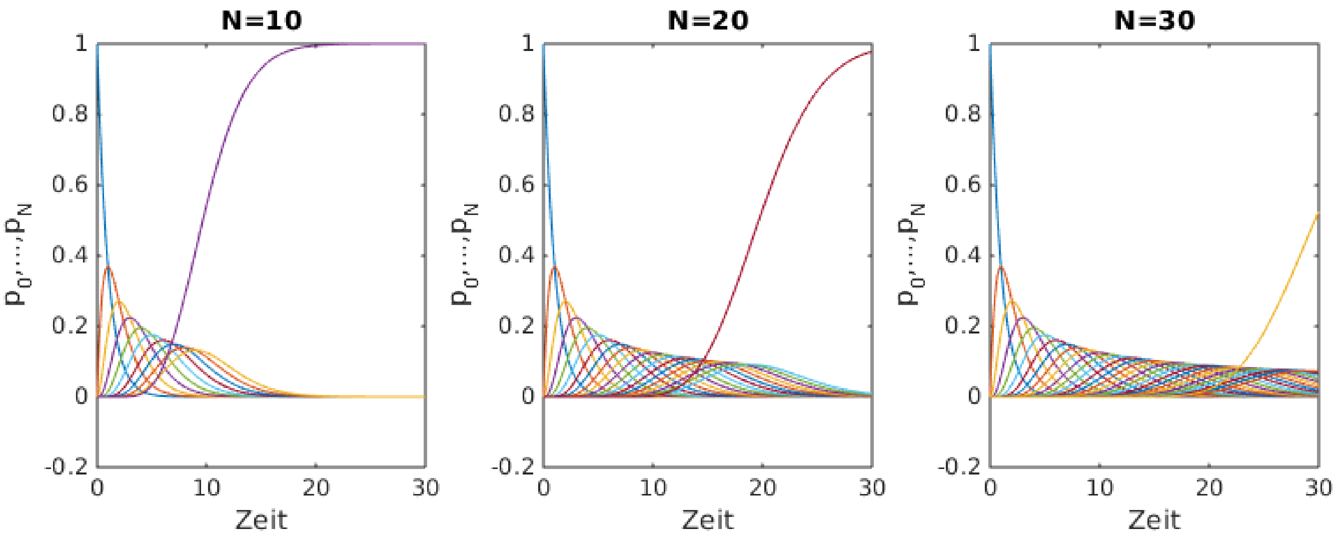
A Matlab function for the solution to this system of ordinary differential equations is given as follows.

```
function [t,p] = treas(N,b,T)

% solve the system of ODEs
%    P'(t) = b*D*P(t),  P(t) = {p_{i+1}(t): i=0,N}
%    p_0(0) = 1, p_i(0) = 0, i=1,...,N

% coefficient matrix
    N1 = N+1;
    D  = spdiags(ones(N1,1),-1,N1,N1) ...
       - spdiags(ones(N1,1), 0,N1,N1);
    D(N1,N1) = 0;

% right side for the system
    dp = @(t,p) b*(D*p);

% solution time interval
    tpte = [0;T];
% options
    opts = odeset('RelTol',1.0e-6);
% initial conditions
    p0   = zeros(N1,1); p0(1) = 1;
% solution
    [t,p] = ode45(dp,tpte,p0,opts);

end
```

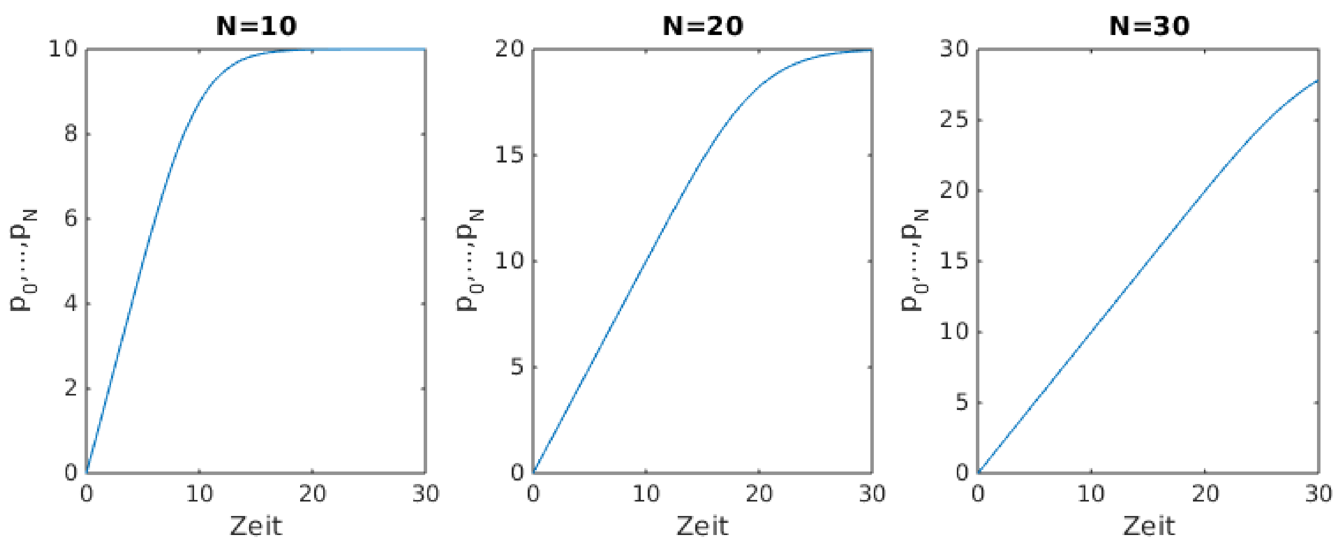
This is called from the following Matlab code to solve the system for various values of N.

```
function S01Exp3

    b = 1;
    T = 30;

    N = 10;
    [t1,p1] = treas(N,b,T);
    N = 20;
    [t2,p2] = treas(N,b,T);
    N = 30;
    [t3,p3] = treas(N,b,T);

    h1 = figure(1); close(h1); h1 = figure(1);
    set(h1,'Position',[10 10 900 300]);
    subplot(1,3,1)
    plot(t1,p1)
    xlabel('Zeit'); ylabel('p_0,...,p_N'); title('N=10')
    subplot(1,3,2)
    plot(t2,p2)
    xlabel('Zeit'); ylabel('p_0,...,p_N'); title('N=20')
    subplot(1,3,3)
    plot(t3,p3)
    xlabel('Zeit'); ylabel('p_0,...,p_N'); title('N=30')

    h2 = figure(2); close(h2); h2 = figure(2);
    set(h2,'Position',[10 10 900 300]);
    subplot(1,3,1)
    plot(t1,p1*(0:10)')
    xlabel('Zeit'); ylabel('p_0,...,p_N'); title('N=10')
    subplot(1,3,2)
    plot(t2,p2*(0:20)')
    xlabel('Zeit'); ylabel('p_0,...,p_N'); title('N=20')
    subplot(1,3,3)
    plot(t3,p3*(0:30)')
    xlabel('Zeit'); ylabel('p_0,...,p_N'); title('N=30')

end
```

Results for $\beta = 1$, $T = 30$ and for $N = 10, 20, 30$ are given first in terms of the following probabilities $\{p_i(t) : t \in [0, T]\}_{i=0}^{N}$



and then in terms of the expected values $\{E(t) : t \in [0, T]\}$, $E(t) = \sum_{i=0}^{N} i p_i(t)$,



In each case $p_i(t)$ reaches a maximum later than $p_{i-1}(t)$ for $1 \leq i \leq N - 1$. Toward the end of the time interval $p_N(t)$ approaches 1, but for N ever larger, $p_N(t)$ approaches 1 ever later. If the system were infinity large (i.e., $N \to \infty$), then $p_i(t) \to 0$, $t \to \infty$, would hold for every $i \geq 0$. For N ever larger the curve $E(t)$ is ever closer to βt.

• Exercise 4: PCA and ICA

◦ Task

For given data,

$$Y = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

(a) carry out PCA to determine Y_c, $K = \frac{1}{4} Y_\mathrm{c} Y_\mathrm{c}^\top$, V and $\Lambda = \mathrm{diag}\{\lambda_i\}$ with $KV = V\Lambda$ and $Y_\mathrm{s} = \Lambda^{-\frac{1}{2}} V^\top Y_\mathrm{c}$. Plot Y and Y_s.

(b) Show that the sphered data satisfy $\frac{1}{4} Y_\mathrm{s} Y_\mathrm{s}^\mathrm{T} = I$. Show that when the sphered data are projected onto an arbitrary axis $\hat{\boldsymbol{w}} \in \mathbb{R}^2$ with $\|\hat{\boldsymbol{w}}\|_{\ell_2} = 1$, the projected data have die variance 1.

(c) For $\theta \in [0, 2\pi]$ define $\boldsymbol{u} = (\cos(\theta), \sin(\theta))$, $\boldsymbol{u}^\perp = (-\sin(\theta), \cos(\theta))$ and $U(\theta) = (\boldsymbol{u}(\theta); \boldsymbol{u}^\perp(\theta)$. Plot $J(\theta) = \mathcal{K}^2(\boldsymbol{u}(\theta) Y_\mathrm{s})$ where $\mathcal{K}$ is the kurtosis. Using this plot, determine the value θ^* which maximizes J. Set $X_\mathrm{c} = U(\theta^*) Y_\mathrm{s}$. Assume that the columns of Y have equal probability, and show that the coordinates (x, y) of columns of the resulting X_c are statistically independent, i.e.,

$$P(x = \alpha \;\&\; y = \beta) = P(x = \alpha) \cdot P(y = \beta).$$

◦ Solution

∗ Part (a) Principle Components

The averaged data are

$$\bar{Y} = \tfrac{1}{4} Y \mathbf{1} = \begin{bmatrix} (0+1+2+3)/4 \\ (0+1+1+2)/4 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}, \quad \mathbf{1} \in \mathbb{R}^4$$

The centered data are

$$Y_\mathrm{z} = Y - \bar{Y}\mathbf{1}^\top = \begin{bmatrix} -1.5 & -0.5 & +0.5 & +1.5 \\ -1.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

The covariance matrix is

$$K = \tfrac{1}{4} Y_\mathrm{z} Y_\mathrm{z}^\top = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 0.50 \end{bmatrix}$$

The eigenspace decomposition of the covariance matrix is

$$KV = V\Lambda, \quad V = \begin{bmatrix} +0.5257 & -0.8507 \\ -0.8507 & -0.5257 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0.0365 & 0 \\ 0 & 1.7135 \end{bmatrix}$$

The sphered data are

$$Y_\mathrm{s} = \Lambda^{-\frac{1}{2}} V^\top Y_\mathrm{z} = \begin{bmatrix} 0.3249 & -1.3764 & 1.3764 & -0.3249 \\ 1.3764 & 0.3249 & -0.3249 & -1.3764 \end{bmatrix}, \quad \text{die erfüllen} \quad \frac{1}{4} Y_\mathrm{s} Y_\mathrm{s}^\top = I$$

∗ **Part (b) Projections**

In general it holds that

$$\begin{aligned}\tfrac{1}{n}Y_\mathrm{s}Y_\mathrm{s}^\top &= \tfrac{1}{n}(\Lambda^{-\frac{1}{2}}V^\top Y_\mathrm{z})(\Lambda^{-\frac{1}{2}}V^\top Y_\mathrm{z})^\top = \Lambda^{-\frac{1}{2}}V^\top(\tfrac{1}{n}Y_\mathrm{z}Y_\mathrm{z}^\top)V\Lambda^{-\frac{1}{2}}\\ &= \Lambda^{-\frac{1}{2}}V^\top KV\Lambda^{-\frac{1}{2}} = \Lambda^{-\frac{1}{2}}V^\top(V\Lambda V^\top)V\Lambda^{-\frac{1}{2}}\\ &= \Lambda^{-\frac{1}{2}}(V^\top V)\Lambda(V^\top V)\Lambda^{-\frac{1}{2}} = \Lambda^{-\frac{1}{2}}\Lambda\Lambda^{-\frac{1}{2}} = I.\end{aligned}$$

For a $\boldsymbol{w} \in \mathbb{R}^2$ with $\|\boldsymbol{w}\|_{\ell_2} = 1$ the projections $\boldsymbol{p} = \boldsymbol{w}\boldsymbol{y}_\mathrm{p}^\top$ of the sphered data Y_s onto the $\boldsymbol{w}$-axis satisfy

$$0 = \boldsymbol{w}^\top(\boldsymbol{p} - Y_\mathrm{s}) = (\boldsymbol{w}^\top\boldsymbol{w})\boldsymbol{y}_\mathrm{p}^\top - \boldsymbol{w}^\top Y_\mathrm{s} = \boldsymbol{y}_\mathrm{p}^\top - \boldsymbol{w}^\top Y_\mathrm{s}.$$

Thus $\boldsymbol{p} = \boldsymbol{w}\boldsymbol{w}^\top Y_\mathrm{s}$ holds, and the coordinates of these points on the $\boldsymbol{w}$-axis are

$$\boldsymbol{y}_\mathrm{p} = \boldsymbol{w}^\top Y_\mathrm{s}$$

The mean and variance of these values are

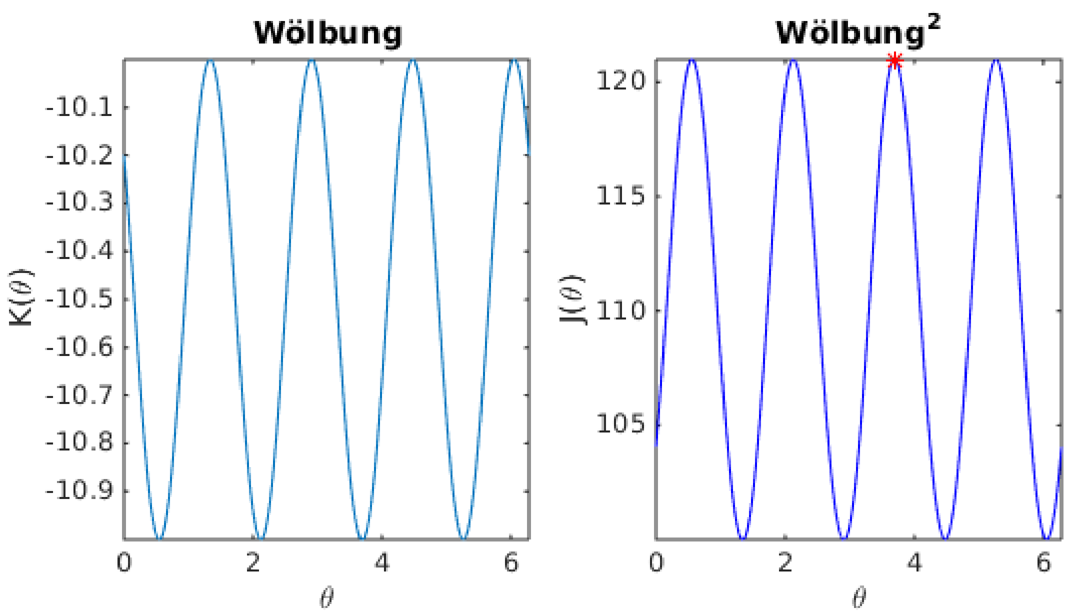
$$\tfrac{1}{4}\boldsymbol{y}_\mathrm{p}\mathbf{1} = \tfrac{1}{4}\boldsymbol{w}^\top Y_\mathrm{s}\mathbf{1} = \tfrac{1}{4}\boldsymbol{w}^\top(\Lambda^{-\frac{1}{2}}V^\top Y_\mathrm{z})\mathbf{1} = \tfrac{1}{4}\boldsymbol{w}^\top\Lambda^{-\frac{1}{2}}V^\top[(Y - \bar{Y}\mathbf{1}^\top)\mathbf{1}] = \boldsymbol{w}^\top\Lambda^{-\frac{1}{2}}V^\top[\tfrac{1}{4}Y\mathbf{1} - \bar{Y}] = 0$$

respectively

$$\tfrac{1}{4}\boldsymbol{y}_\mathrm{p}\boldsymbol{y}_\mathrm{p}^\top = \tfrac{1}{4}(\boldsymbol{w}^\top Y_\mathrm{s})(\boldsymbol{w}^\top Y_\mathrm{s})^\top = \boldsymbol{w}^\top(\tfrac{1}{4}Y_\mathrm{s}Y_\mathrm{s})^\top)\boldsymbol{w} = \boldsymbol{w}^\top\boldsymbol{w} = 1.$$

∗ **Part (c) Independent Components**

The graphical representation of the functions $\mathcal{K}(\theta) = \mathcal{K}(\boldsymbol{u}(\theta)Y_\mathrm{s})$ respectively $J(\theta) = \mathcal{K}^2(\boldsymbol{u}(\theta)Y_\mathrm{s})$ are



where a red ∗ marks the maximum. The maximizing value for $J(\theta)$ is

$$\theta^\star = 3.6945$$

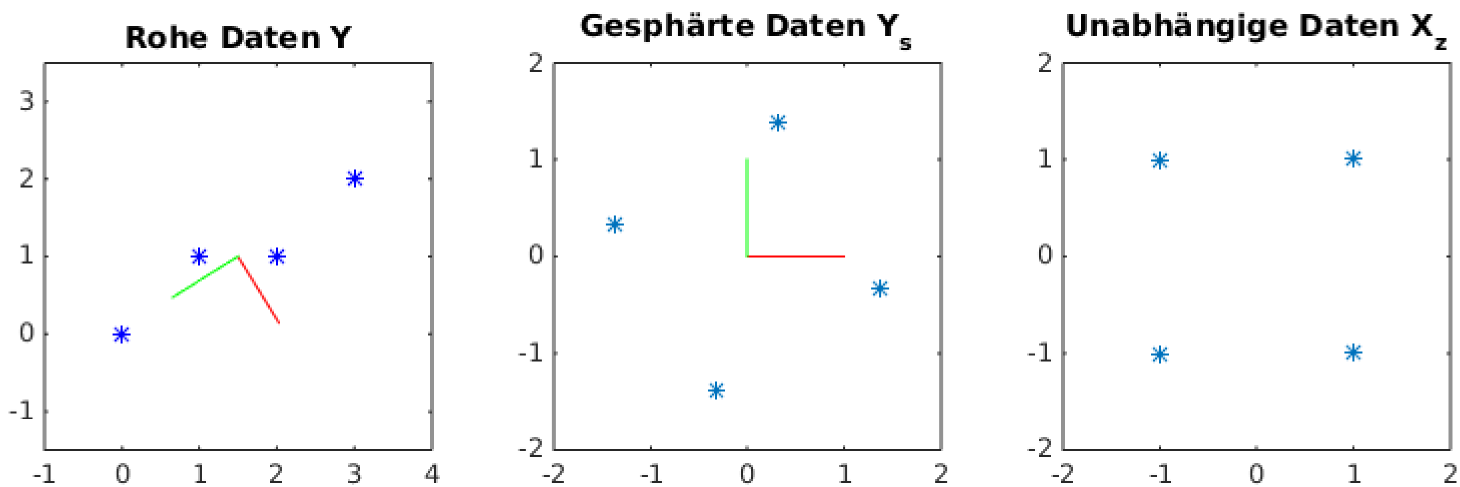
and the maximally independet data are

$$X_\mathrm{z} = U(\theta^\star)Y_\mathrm{s} = \begin{bmatrix} -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +1 \end{bmatrix}$$

where each column satisfies the following with equal probability,

$$P(x = \alpha \& y = \beta) = P(x = \alpha) \cdot P(y = \beta), \quad x, y = \pm 1.$$

The graphical representation of the data Y, Y_s and X_z is



These results are computed with the following Matlab Code.

```
Y=[0 1 2 3;0 1 1 2];
Yq = Y*ones(4,1)/4;
Yz = Y-Yq*ones(1,4);
K  = Yz*Yz'/4;
[V,D] = eig(K);
Ys = D^(-1/2)*V'*Yz;
f = @(t) sum(([cos(t),sin(t)]*Ys).^4)/4-3*sum(([cos(t),sin(t)]*Ys).^2)^2/4;
n = 1001;
th = linspace(0,2*pi,n);
fh = zeros(1,n);
for i=1:n
   fh(i) = f(th(i));
end
ih = find(fh.^2 == max(fh.^2),1);
tt = th(ih);
Xz = [cos(tt),sin(tt);-sin(tt),cos(tt)]*Ys;

h2 = figure(2); close(h2); h2 = figure(2);
set(h2,'Position',[10 10 600 300]);
subplot(1,2,1)
plot(th,fh)
xlabel('\theta')
ylabel('K(\theta)')
```

```
axis tight
title('Wlbung')
subplot(1,2,2)
plot(th,fh.^2,'b',tt,max(fh.^2),'r*')
xlabel('\theta')
ylabel('J(\theta)')
axis tight
title('Wlbung^2')

h1 = figure(1); close(h1); h1 = figure(1);
set(h1,'Position',[10 10 900 300]);

subplot(1,3,1)
plot(Y(1,:),Y(2,:),'b*', ...
     [Yq(1),Yq(1)+V(1,1)],[Yq(2),Yq(2)+V(1,2)],'r', ...
     [Yq(1),Yq(1)+V(2,1)],[Yq(2),Yq(2)+V(2,2)],'g')
axis([-1 4 -1.5 3.5])
pbaspect([1 1 1])
title('Rohe Daten Y')

subplot(1,3,2)
plot(Ys(1,:),Ys(2,:),'*',[0,1],[0,0],'r',[0,0],[0,1],'g')
axis([-2 2 -2 2])
pbaspect([1 1 1])
title('Gesphrte Daten Y_s')

subplot(1,3,3)
plot(Xz(1,:),Xz(2,:),'*')
axis([-2 2 -2 2])
pbaspect([1 1 1])
title('Unabhngige Daten X_z')
```

• Exercise 5: Dimensional Analysis

○ Task

With dimensional analysis derive the third Kepler Law: *The squares of the orbital periods of two planets are proportional to the cubes of their semi-major axes.* (Hint: See page 35 in the script `http://imsc.uni-graz.at/thaller/lehre/mpt/skriptum.pdf`, and don't forget to verify the conditions of the Buckingham Pi Theorem.)

○ Solution

∗ Part (a) Stone

The $n = 6$ involved quantities are

G_1	v	impact velocity	LZ^{-1}
G_2	h	height of the stone	L
G_3	m_S	mass of the stone	M
G_4	m_E	mass of the Earth	M
G_5	g	acceleration on Earth surface	LZ^{-2}
G_6	τ	fall time of the stone	Z

with respect to the $r = 3$ base quantities g_1 = length (L), g_2 = time (Z) und g_3 = mass (M). The 2., 3. and 6. columns of the matrix

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}}_{G_1\ G_2\ G_3\ G_4\ \ G_5\ G_6} \begin{matrix} g_1 \\ g_2 \\ g_3 \end{matrix}$$

are clearly linearly independet, and so the matrix A has rank $r = 3$. According to the Buckingham Pi theorem $n - r = 3$ dimensionless combinations $\{\Pi_1, \Pi_2, \Pi_3\}$ of the derived quantities $\{G_1, \ldots, G_6\}$ are sought, where $\Phi(G_1, \ldots, G_6) = 1$ may be rewritten as $\Psi(\Pi_1, \Pi_2, \Pi_3)$. One choses very simply,

$$\Pi_3 = m_\mathrm{S}/m_\mathrm{E}$$

since this can be estimated with a constant, namely $\Pi_3 \approx 0$. Yet 2 more are sought which are independent of this one. Simply trying possibilities, one writes v and supplements it with other quantities until the product is dimensionless,

$$\Pi_1 = v\tau h^{-1}.$$

Since acceleration on the Earth surface is constant, the following can be estimated with a constant,

$$\Pi_2 = g\tau v^{-1}.$$

The quantitites Π_1 and Π_2 are of course not uniquely determined. The desired physical relationship should describe the impact velocity. It is of the form

$$v = F(h, m_\mathrm{S}, m_\mathrm{E}, g, \tau)$$

where the function F is still unknown. Since the vectors

$$\begin{aligned} \Pi_1: \ \boldsymbol{\lambda}_1 &= (1, -1, 0, 0, 0, 1) \\ \Pi_2: \ \boldsymbol{\lambda}_2 &= (-1, 0, 0, 0, 1, 1) \\ \Pi_3: \ \boldsymbol{\lambda}_3 &= (0, 0, 1, -1, 0, 0) \end{aligned}$$

are linearly independent, the Buckingham Pi theorem can be used to replace the relation in F with a relation involving the dimensionless quantities,

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

i.e.,

$$v\tau h^{-1} = f(g\tau v^{-1}, m_\mathrm{S}/m_\mathrm{E})$$

or

$$v = h\tau^{-1} f(g\tau v^{-1}, m_\mathrm{S}/m_\mathrm{E})$$

where the function f is still unknown. Yet the mass of the Earth is very large and with $g \approx$ constant, it follows that $v/\tau \approx g$. There results approximately

$$\Pi_2 = g\tau v^{-1} \approx 1, \quad \Pi_3 = m_{\mathrm{S}}/m_{\mathrm{E}} \approx 0.$$

Thus it is assumed

$$f(\Pi_2, \Pi_3) = f(g\tau v^{-1}, m_{\mathrm{S}}/m_{\mathrm{E}}) \overset{\Pi_2,\Pi_3 \to (1,0)}{\longrightarrow} k \quad \text{(constant)}.$$

It follows

$$v = kh/\tau$$

where the constant can be estimated with $k \approx 2$ through experimentation.

∗ Part (b) Atom Bomb

The $n = 9$ involved quantities are

G_1	E	energy of the atomic bomb	$\mathrm{ML^2Z^{-2}}$
G_2	t	time since the ignition	Z
G_3	R	radius of the fireball	L
G_4	ρ_{A}	outside air density	$\mathrm{ML^{-3}}$
G_5	ρ_{I}	inside air density	$\mathrm{ML^{-3}}$
G_6	p_{A}	outside air pressure	$\mathrm{ML^{-1}Z^{-2}}$
G_7	p_{I}	inside air pressure	$\mathrm{ML^{-1}Z^{-2}}$
G_8	T_{A}	outside air temperature	T
G_9	T_{I}	inside aire temperature	T

with respect to $r = 4$ base quantities $g_1 =$ length (L), $g_2 =$ time (Z), $g_3 =$ mass (M) and $g_4 =$ temperature (T). The 1., 2., 3. and 9. columns of the matrix

$$A = \underbrace{\begin{bmatrix} 2 & 0 & 1 & -3 & -3 & -1 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & -2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{G_1 \; G_2 \; G_3 \; G_4 \; G_5 \; G_6 \; G_7 \; G_8 \; G_9} \begin{matrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{matrix}$$

are clearly linearly independet, and so the matrix A has ranke $r = 3$. According to the Buckingham Pi theorem $n - r = 5$ dimensionless combinations $\{\Pi_1, \Pi_2, \Pi_3\}$ of the derived quantities $\{G_1, \ldots, G_6\}$ are sought, where $\Phi(G_1, \ldots, G_6) = 1$ may be rewritten as $\Psi(\Pi_1, \Pi_2, \Pi_3)$. One choses very simply,

$$\Pi_3 = \rho_{\mathrm{A}}/\rho_{\mathrm{I}}, \quad \Pi_4 = p_{\mathrm{A}}/p_{\mathrm{I}}, \quad \Pi_5 = T_{\mathrm{A}}/T_{\mathrm{I}}$$

since this can be estimated with a constant, namely $\Pi_3, \Pi_4, \Pi_5 \approx 0$. Yet 2 more are sought which are independent of these. Simply trying possibilities, one writes E and supplements it with other quantities until the product is dimensionless,

$$\Pi_1 = E(\rho_{\mathrm{A}}^{-1} R^5 t^2).$$

Compared with p_{A} and t, E und ρ_{I} are large, and therefore

$$\Pi_2 = t^6 p_{\mathrm{A}}^5 E^{-2} \rho_{\mathrm{I}}^{-3}$$

can be well estimated with a constant, $\Pi_2 \approx 0$. The quantitites Π_1 and Π_2 are of course not uniquely determined. The desired physical relationship should describe the energy. It is of the form

$$E = F(t, R, \rho_{\mathrm{A}}, \rho_{\mathrm{I}}, p_{\mathrm{A}}, p_{\mathrm{I}}, T_{\mathrm{A}}, T_{\mathrm{I}})$$

where the function F is still unknown. Since the vectors

$$\begin{array}{rcl} \Pi_1: \ \boldsymbol{\lambda}_1 & = & (1, 2, -5, -1, 0, 0, 0, 0, 0) \\ \Pi_2: \ \boldsymbol{\lambda}_2 & = & (-2, 6, 0, 0, -3, 5, 0, 0, 0) \\ \Pi_3: \ \boldsymbol{\lambda}_3 & = & (0, 0, 0, 1, -1, 0, 0, 0, 0) \\ \Pi_4: \ \boldsymbol{\lambda}_4 & = & (0, 0, 0, 0, 0, 1, -1, 0, 0) \\ \Pi_5: \ \boldsymbol{\lambda}_4 & = & (0, 0, 0, 0, 0, 0, 0, 1, -1) \end{array}$$

are linearly independent, the Buckingham Pi theorem can be used to replace the relation in F with a relation involving the dimensionless quantities,

$$\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4, \Pi_5)$$

i.e.,

$$E(\rho_{\mathrm{A}}^{-1} R^5 t^2) = f(t^6 p_{\mathrm{A}}^5 E^{-2} \rho_{\mathrm{I}}^{-3}, \rho_{\mathrm{A}}/\rho_{\mathrm{I}}, p_{\mathrm{A}}/p_{\mathrm{I}}, T_{\mathrm{A}}/T_{\mathrm{I}})$$

or

$$E = (\rho_{\mathrm{A}} R^{-5} t^{-2}) f(t^6 p_{\mathrm{A}}^5 E^{-2} \rho_{\mathrm{I}}^{-3}, \rho_{\mathrm{A}}/\rho_{\mathrm{I}}, p_{\mathrm{A}}/p_{\mathrm{I}}, T_{\mathrm{A}}/T_{\mathrm{I}})$$

where the function f is still unknown. Yet through

$$\Pi_2, \Pi_3, \Pi_4, \Pi_5 \approx 0$$

it is assumed that f can be estimated with a constant,

$$f(\Pi_2, \Pi_3, \Pi_4, \Pi_5) = f(t^6 p_{\mathrm{A}}^5 E^{-2} \rho_{\mathrm{I}}^{-3}, \rho_{\mathrm{A}}/\rho_{\mathrm{I}}, p_{\mathrm{A}}/p_{\mathrm{I}}, T_{\mathrm{A}}/T_{\mathrm{I}}) \overset{\Pi_2,\Pi_3,\Pi_4,\Pi_5 \to 0}{\longrightarrow} k \quad \text{(constant)}.$$

It follows

$$E = k \rho_{\mathrm{A}} R^{-5} t^{-2}$$

where the constant k must be estimated through experimentation. With a video of the explosion, the data $\{(t_n, R_n)\}_{n=1}^N$ (fireball radius vs. time) are available. Through

$$R^5 = \left(\frac{E}{\rho_{\mathrm{A}} k}\right) t^2 \quad \text{oder} \quad R(t) = \gamma t^{2/5}$$

the constant

$$\gamma = \left(\frac{E}{\rho_{\mathrm{A}} k}\right)^{\frac{1}{5}} \quad \text{is so estimated:} \quad \gamma \approx \frac{1}{N} \sum_{n=1}^{N} R_n t_n^{-2/5}.$$

One carrys out a sufficiently similar experiment, in which a known about of explosive mit known energy E_0 is ignited. For this it is assumed that the above constant k is again the same. For the course of the controlled explosion the data $\{(t_0, R_0)_m\}_{m=1}^M$ (fireball radius vs. time) are measured and the constant is calculated,

$$\gamma_0 = \left(\frac{E_0}{\rho_{\mathrm{A}} k}\right)^{\frac{1}{5}} \quad \text{so abgeschätzt:} \quad \gamma_0 \approx \frac{1}{M} \sum_{m=1}^{M} (R_0 t_0^{-2/5})_m.$$

It follows that the sought energy E is given through $k = E/(\rho_{\mathrm{A}} \gamma^5) = E_0/(\rho_{\mathrm{A}} \gamma_0^5)$ or

$$E = E_0 (\gamma/\gamma_0)^5.$$

* **Part (c) Kepler's Law**

(The solution in the lectures notes of Prof Thaler should be supplemented with the confirmation that the prerequisites of the Buckingham Pi theorem are satisfied.) First the list of $n = 6$ involved quantities is:

G_1	τ	orbital period	Z
G_2	G	gravitational constant	$\mathrm{M}^{-1}\mathrm{L}^3\,\mathrm{Z}^{-2}$
G_3	m_S	mass of the sun	M
G_4	m_P	mass of the planet	M
G_5	r	length of the major axis	L
G_6	e	eccentricity	1

with respect to the $r = 3$ base quantities g_1 = length (L), g_2 = time (Z) and g_3 = mass (M). According to Keplers first law the planet trajectories are ellipses. Therefore the eccentricity is included in the list. The eccentricity e of an ellipse is the distance from the center to the focus divided by the length of the major axis, so it is dimensionless. The 1., 3. and 5. columns of the matrix

$$A = \underbrace{\begin{bmatrix} 0 & 3 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{bmatrix}}_{G_1\ G_2\ G_3\ G_4\ G_5\ G_6} \begin{matrix} g_1 \\ g_2 \\ g_3 \end{matrix}$$

are clearly linearly independent, so the matrix A has ranke $r = 3$. According to the Buckingham Pi theorem, $n - r = 3$ dimensionless combinations $\{\Pi_1, \Pi_2, \Pi_3\}$ of the derived quantities Größen $\{G_1, \ldots, G_6\}$ are sought, where $\Phi(G_1, \ldots, G_6) = 1$ may be rewritten as $\Psi(\Pi_1, \Pi_2, \Pi_3)$. One choses very simply,

$$\Pi_2 = e, \quad \Pi_3 = m_\mathrm{P}/m_\mathrm{S}$$

since this can be estimated with a constant, namely $\Pi_2, \Pi_3 \approx 0$. Yet another is sought which is independent of these. Simply trying possibilities, one writes G and supplements it with other quantities until the product is dimensionless,

$$\Pi_1 = G m_\mathrm{S} \tau^2 r^{-3}$$

This choice for Π_1 is of course not uniquely determined. The desired physical relationship should describe the orbital period. It is of the form

$$\tau = F(G, m_\mathrm{P}, m_\mathrm{S}, r, e)$$

where the function F is still unknown. Since the vectors

$$\begin{aligned} \Pi_1 : \ \boldsymbol{\lambda}_1 &= (2, 1, 1, 0, -3, 0) \\ \Pi_2 : \ \boldsymbol{\lambda}_2 &= (0, 0, 0, 0, 0, 1) \\ \Pi_3 : \ \boldsymbol{\lambda}_3 &= (0, 0, -1, 1, 0, 0) \end{aligned}$$

are linearly independent, the Buckingham Pi theorem can be used to replace the relation in F with a relation involving the dimensionless quantities,

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

i.e.,

$$G m_\mathrm{S} \tau^2 r^{-3} = f(e, m_\mathrm{P}/m_\mathrm{S})$$

or

$$\tau^2 = r^3 G m_\mathrm{S} f(e, m_\mathrm{P}/m_\mathrm{S})$$

where the function f is still unknown. One considers that in real planetary systems the eccentricity of the orbits is very small, and the mass of the sun is very large, so it is a good approximation that

$$\Pi_2 = e \approx 0, \quad \Pi_3 = m_\mathrm{P}/m_\mathrm{S} \approx 0.$$

So it is assumed that

$$f(\Pi_2, \Pi_3) = f(e, m_\mathrm{P}/m_\mathrm{S}) \overset{\Pi_2,\Pi_3 \to 0}{\longrightarrow} k \quad \text{(constant)}.$$

This assumption should not be taken for granted. An arbitrary function f need not to be continuous or even bounded near a chosen point. In physics one simply often tries to push an argument as far as possible with very few assumptions. There now results

$$\tau^2 = k G m_\mathrm{S} r^3.$$

Thus the square of the orbital time is proportional to the third power of the distance to the sun. Since the constant of proportionality $k/(G m_\mathrm{S})$ can be assumed to be independent of the planet considered, there results the third Kepler law when the ratio τ_1^2/τ_2^2 is considered for two planets. By solving Newton's equations for planetary systems, there results for the function f,

$$f(e, m_\mathrm{P}/m_\mathrm{S}) = 4\pi^2.$$

So the eccentricity and the mass of the planet are not even involved, and the correct formula is,

$$\tau^2 = 4\pi^2 G m_\mathrm{S} r^3.$$

It is noteworthy that this formula has been found without computational expense. The method requires only an informed understanding of physics, since this is necessary to identify the relevant physical quantities. If m_S were taken, e.g., to be the mass of Sirius and r were taken to be the distance between the planet and the center of the galaxy, then the result may reflect correct dimensions and yet be senseless and physically incorrect.

• Exercise 6: Chemical Reactions

○ Task

The chemical reaction

$$kA + lB \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} mC + nD$$

is given with the following parameters:

$$k = 2, \quad l = 1, \quad m = 2, \quad n = 1$$

$$[A](0) = 2, \quad [B](0) = 2, \quad [C](0) = 2, \quad [D](0) = 1.$$

Determine the initial value problem

$$x'(t) = f(x(t); \kappa_1, \kappa_2), \quad x(0) = x_0$$

where

$$[A](t) = [A](0) - kx(t), \quad [B](t) = [B](0) - lx(t)$$

$$[C](t) = [C](0) + mx(t), \quad [D](t) = [D](0) + nx(t).$$

Rewrite the equation $f(x^*; \kappa_1, \kappa_2) = 0$ in the form,

$$\kappa_2/\kappa_1 = r(x^\star)$$

giving a relationship between equilibria and the quotient κ_2/κ_1. (Note that the condition $\kappa_2/\kappa_1 > 0$ implies, through the form of r, that an equilibrium must satisfy $x^\star > -1$.) Show that any equilibrium $x^\star$ with $r'(x^\star) < 0$ is (locally asymptotically) stable, while any equilibrium $x^\star$ with $r'(x^\star) > 0$ is unstable. (Hint: It holds that $f'(x^\star; \kappa_1, \kappa_2)/\kappa_1 = 4r'(x^\star)(1 + x^\star)^3$.) Derive a corresponding potential landscape $p(x, \kappa_2/\kappa_1)$, where $f(x; \kappa_1, \kappa_2)/\kappa_1 = -p_x(x, \kappa_2/\kappa_1)$. For various values of κ_2/κ_1, plot $f(x; \kappa_1, \kappa_2)$ and $p(x, \kappa_2/\kappa_1)$ in the interval $0 \le x \le 3$ to demonstrate the associated Hysteresis graphically.

○ Solution

The initial value problem for the reaction is

$$x'(t) = f(x), \quad x(0) = 0$$

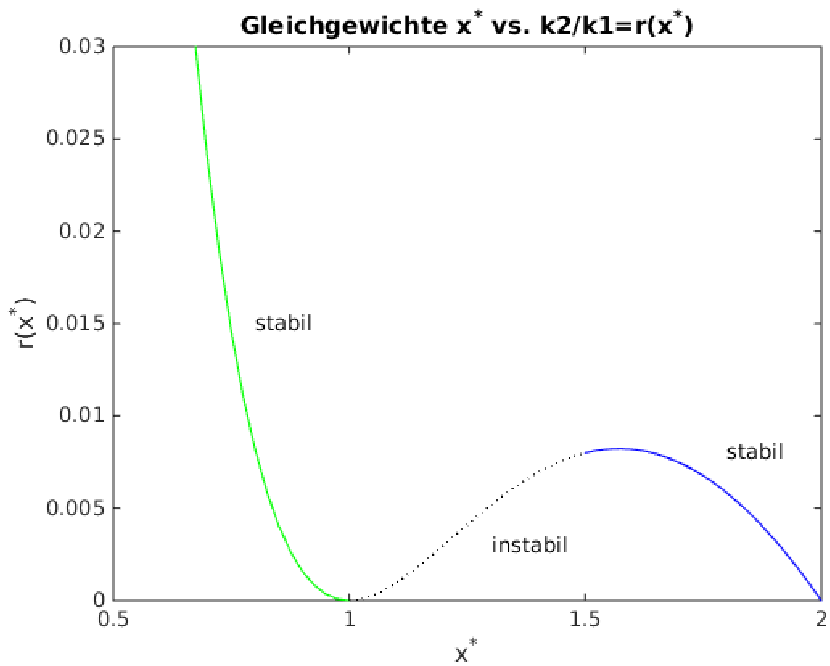
where

$$f(x) = \kappa_1(2 - 2x)^2(2 - x) - \kappa_2(1 + x)(2 + 2x)^2.$$

An equilibrium $x^\star$ for the reaction satisfies $f(x^\star) = 0$ or

$$0 < \frac{\kappa_2}{\kappa_1} = \frac{(1 - x^\star)^2(2 - x^\star)}{(1 + x^\star)^3} =: r(x^\star) \quad \text{für } x^\star \in (-1, 2)$$

where $r(x)$ has the following graphical representation.



It remains to show that the equilibrium states $x^\star$ satisfying

$$r'(x^\star) < 0$$

are (locally asymptotically) stable, while the equilibrium states $x^\star$ satisfying

$$r'(x^\star) > 0$$

are unstable. Due to the calculation

$$r'(x) = \frac{-11 + 18x - 7x^2}{(1+x)^4} = \frac{(x-1)(11-7x)}{(1+x)^4}$$

it follows

$$r'(x) < 0, \quad x \in (-1,1) \cup (11/7, +\infty) \quad \text{and} \quad r'(x) > 0, \quad x \in (1, 11/7)$$

Due to the calculation

$$f'(x) = -4\left[3\kappa_2(1+x)^2 + \kappa_1(5 - 8x + 3x^2)\right] = -4\left[3\frac{\kappa_2}{\kappa_1}(1+x)^2 + (5 - 8x + 3x^2)\right]\kappa_1$$

it follows with $\kappa_2/\kappa_1 = r(x^\star)$,

$$\frac{f'(x^\star)}{4\kappa_1} = 3r(x^\star)(1+x^\star)^2 + (5 - 8x^\star + 3x^{\star 2}) = \frac{-11 + 18x^\star - 7x^{\star 2}}{(1+x^\star)} = r'(x^\star)(1+x^\star)^3$$

and the claimed stability follows:

$$f'(x^\star) < 0, \quad x^\star \in (-1,1) \cup (11/7, +\infty) \quad \text{und} \quad f'(x^\star) > 0, \quad x^\star \in (1, 11/7)$$

With

$$p(x, R) = -8x + 10x^2 - 16x^3/3 + x^4 + R(1+x)^4$$

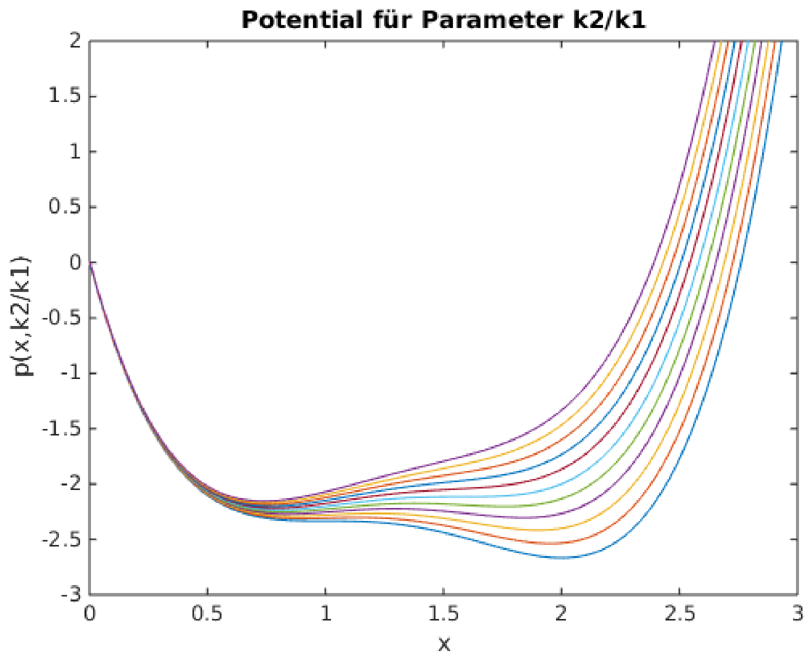
follows

$$f(x)/\kappa_1 = -p_x(x, \kappa_2/\kappa_1)$$

and p is a potential for the function f. For the values

$$R = \frac{2k}{10} \cdot r(11/7) = \frac{2k}{10} \cdot \max_{1 \le x \le 2} r(x), \quad k = 0, \ldots, 10$$

the potential landscape appears as follows,



where the index k increases through these curves from the bottom to the top. The deepest point at the left corresponds to a stable equilibrium $x^\star > 11/7$ where $r'(x^\star) < 0$ holds. The highest point in the middle corresponds to an unstable equilibrium $x^\star \in (1, 11/7)$ where $r'(x^\star) > 0$ holds. As R increases from 0 to $2 \cdot r(11/7)$, the solution remains at the equilibrium in the right valley, as long as it is a valley, even if the left valley is deeper. As soon as the deep point to the right is no longer in a valley, the equilibrium jumps into the valley to the left. On the other hand, as R decreases from $2 \cdot r(11/7)$ to 0, the solution remains at an equilibrium in the left valley, as long as it is a valley, even if the right valley is deeper. As soon as the deep point to the left is no longer in a valley, the equilbrium jumps into the valley to the right. In this way the system exhibits hysteresis.

• Exercise 7: Chemical Kinetics and Hysteresis

○ Task

Show that the equilibrium $(x^\star, y^\star) = (a_2/b_2, a_1/b_1)$ for the Gause equations,

$$x'(t) = (a_1 - b_1 y)x, \quad y'(t) = (a_2 - b_2 x)y, \quad a_1, a_2, b_1, b_2 > 0$$

is unstable and, in particular, a saddle point.

○ Solution

Let $(x(t), y(t))$ be a solution curve. It holds then

$$D_t Q(x(t), y(t)) = -a_2 x'(t)/x(t) + b_2 x'(t) + a_1 y'(t)/y(t) - b_1 y'(t)$$

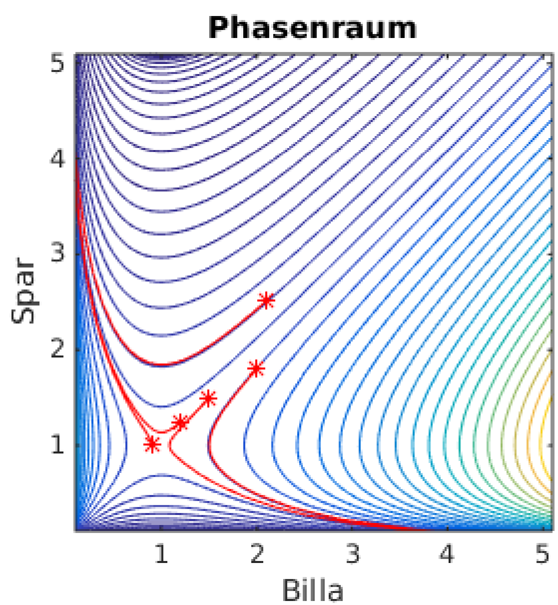
$$= (a_1 - b_1 y(t))(b_2 x(t) - a_2) + (a_2 - b_2 x(t))(a_1 - b_1 y(t)) = 0$$

and therefore Q remains constant on the solution curve, i.e., the solution curve lies in a niveau-curve of the the function Q. A Matlab code for the graphical representation of the niveau-curves of Q is given as follows.

```
% setup figure
    h1 = figure(1); close(h1); h1 = figure(1);
    set(h1,'Position',[10 10 300 300]);

% Parameter
    a1 = 1; b1 = 1; a2 = 1; b2 = 1;

% Gause Modell
    gause = @(t,X) [(a1 - b1*X(2))*X(1);(a2 - b2*X(1))*X(2)];

% Gitter fuer Niveau Kurven im Phasenraum
    n = 101;
    xmin = 1.0e-1; xmax = 5.1e0; ymin = 1.0e-1; ymax = 5.1e0;
    x = linspace(xmin,xmax,n)';
    y = linspace(ymin,ymax,n);
    xx = kron(x,ones(size(y)));
    yy = kron(ones(size(x)),y);

% Auswertung von Q auf dem Gitter
    zz = -a2*log(xx) + b2*xx + a1*log(yy) - b1*yy;
    zz = exp(zz);

    zmin = min(zz(:));
    zmax = max(zz(:));

% Verteilung von Niveau-Kurven
    m = 50;
    w = zmin + (zmax - zmin)*((0:(m-1))/(m-1)).^5;

% grafische Darstellung der Niveau-Kurven
    contour(xx,yy,zz,w);
    axis([xmin xmax ymin ymax])
    pbaspect([1 1 1]);

    xlabel('Billa')
    ylabel('Spar');
    title('Phasenraum');

% Untersuchung der Stabilitaet des Gleichgewichts (a2/b2,a1/b1)
    hold on;
    tspan = [0,10];

    X0 = [a2/b2;a1/b1] + [1.1;1.51];
    [t,X] = ode15s(gause,tspan,X0);
```

```
plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

X0 = [a2/b2;a1/b1] + [1;0.8];
[t,X] = ode15s(gause,tspan,X0);
plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

X0 = [a2/b2;a1/b1] + [0.5;0.49];
[t,X] = ode15s(gause,tspan,X0);
plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

X0 = [a2/b2;a1/b1] + [0.2;0.245];
[t,X] = ode15s(gause,tspan,X0);
plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

X0 = [a2/b2;a1/b1] + [-0.1;0];
[t,X] = ode15s(gause,tspan,X0);
plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

hold off;
```

The result is as follows.



Since the red curves keep a distance from the equilibrium, the equilibrium is unstable according to these calculations. This instability should be shown theoretically. First with $\boldsymbol{x} = (x, y)$ and $\boldsymbol{f}(\boldsymbol{x}) = ((a_1 - b_1 y)x, (a_2 - b_2 x)y)$ the Gause system can be rewritten with $\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t))$. It follows

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(x,y) = \begin{bmatrix} (a_1 - b_1 y) & -b_1 x \\ -b_2 y & (a_2 - b_2 x) \end{bmatrix}, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(a_2/b_2, a_1/b_1) = \begin{bmatrix} 0 & -b_1 a_2/b_2 \\ -b_2 a_1/b_1 & 0 \end{bmatrix}$$

and the eigenvalues of the matrix on the right are $\pm\sqrt{a_1 a_2}$. Since there a positive eigenvalue, the equilibrium is unstable.

• Exercise 8: Mass Spring Model

◦ Task

Let the damped mass spring model,

$$mu''(t) = f - ku(t) - cu'(t)$$

for $m = 1$, $k = 1$, $f = 1$ and $c = 3$ be written in first order form

$$\boldsymbol{u}'(t) = A\boldsymbol{u}(t) + \boldsymbol{b}, \quad \boldsymbol{u}(t) = (u(t); u'(t)), \quad A = [0, 1; -k, -c]/m, \quad \boldsymbol{b} = (0; f/m).$$

First show that the equilibrium

$$\boldsymbol{u}^\star = (u^\star; 0) = -A^{-1}\boldsymbol{b}$$

is locally asymptotically stable. Then note that the system can also be written as

$$\boldsymbol{w}'(t) = A\boldsymbol{w}(t), \quad \boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^\star.$$

To show that $\boldsymbol{w} = \boldsymbol{0}$ is globally asymptotically stable, derive a function $P(\boldsymbol{w})$ satisfying

$$S^{-\top}S^{-1}A\boldsymbol{w} = -\nabla P(\boldsymbol{w}), \quad AS = S\Lambda, \quad \Lambda = \text{diag}\{\lambda_i\}_{i=1}^2$$

and show that P decreases to its global minimum at $\boldsymbol{w} = \boldsymbol{0}$ along every solution $\boldsymbol{w}(t)$.

◦ Solution

For the system,

$$\left\{ \begin{array}{rcll} \boldsymbol{u}'(t) & = & A\boldsymbol{u}(t) + \boldsymbol{b}, & t > 0 \\ \boldsymbol{u}(0) & = & \boldsymbol{u}_0, & t = 0 \end{array} \right. \quad \boldsymbol{u}(t) = \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}, \quad \boldsymbol{u}_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$A = \frac{1}{m}\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}, \quad \boldsymbol{b} = \frac{1}{m}\begin{bmatrix} 0 \\ f \end{bmatrix}.$$

the eigenspace decomposition of the matrix A is given by

$$A = S\Lambda S^{-1}, \quad \Lambda = -\frac{1}{2m}\begin{bmatrix} c + \sqrt{c^2 - 4k} & 0 \\ 0 & c - \sqrt{c^2 - 4k} \end{bmatrix}$$

$$S = -\frac{1}{2k}\begin{bmatrix} c - \sqrt{c^2 - 4k} & c + \sqrt{c^2 - 4k} \\ -2k & -2k \end{bmatrix}, \quad S^{-1} = \frac{1}{2\sqrt{c^2 - 4k}}\begin{bmatrix} 2k & c + \sqrt{c^2 - 4k} \\ -2k & -c + \sqrt{c^2 - 4k} \end{bmatrix}$$

and in particular for the values $m = 1$, $k = 1$, $f = 1$ and $c = 3$,

$$A = S\Lambda S^{-1}, \quad \Lambda = -\frac{1}{2}\begin{bmatrix} 3 + \sqrt{5} & 0 \\ 0 & 3 - \sqrt{5} \end{bmatrix}$$

$$S = -\frac{1}{2}\begin{bmatrix} 3 - \sqrt{5} & 3 + \sqrt{5} \\ -2 & -2 \end{bmatrix}, \quad S^{-1} = \frac{1}{2\sqrt{5}}\begin{bmatrix} 2 & 3 + \sqrt{5} \\ -2 & -3 + \sqrt{5} \end{bmatrix}$$

and the eigenvalues $\lambda_{\text{min}} = -(3-\sqrt{5})/2$ and $\lambda_{\text{max}} = -(3+\sqrt{5})/2$ are negative. Thus A is non-singular, and the equilibrium

$$\boldsymbol{u}^\star = \begin{bmatrix} u^\star \\ 0 \end{bmatrix} = -A^{-1}\boldsymbol{b}, \quad u^\star = \frac{f}{k} = 1$$

is well defined. Since the eigenvalues are negative, the equilibrium is locally asymptotically stable.

Then the vector $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^\star$ satisfies

$$\boldsymbol{w}'(t) = D_t[\boldsymbol{u}(t) - \boldsymbol{u}^\star] = \boldsymbol{u}'(t) = A\boldsymbol{u}(t) + \boldsymbol{b} = A\boldsymbol{u}(t) - A\boldsymbol{u}^\star = A\boldsymbol{w}(t)$$

$$\boldsymbol{w}(0) = \boldsymbol{u}(0) - \boldsymbol{u}^\star = \boldsymbol{u}_0 - \boldsymbol{u}^\star = \boldsymbol{w}_0$$

and $\boldsymbol{w}^\star = \boldsymbol{0}$ is the only equilibrium. The matrix $S^{-\top}S^{-1}A$ is given by

$$-S^{-\top}S^{-1}A = \frac{1}{m(c^2-4k)}\begin{bmatrix} ck^2 & k(c^2-2k) \\ k(c^2-2k) & c(c^2-3k) \end{bmatrix}$$

and in particular for the values $m = 1$, $k = 1$, $f = 1$ and $c = 3$,

$$-S^{-\top}S^{-1}A = \frac{1}{5}\begin{bmatrix} 3 & 7 \\ 7 & 18 \end{bmatrix} = H.$$

So that the matrix H agrees with the Hessian of a necessarily quadratic function $P(w_1, w_2) = aw_1^2 + 2bw_1w_2 + cw_2^2$, it must hold that

$$2a = P_{w_1,w_1} = \frac{3}{5}, \quad 2b = P_{w_1,w_2} = \frac{7}{5}, \quad 2c = P_{w_2,w_2} = \frac{18}{5}$$

or

$$P(w_1, w_2) = \frac{3w_1^2 + 14w_1w_2 + 18w_2^2}{10} = \frac{1}{2}\boldsymbol{w}^\top H\boldsymbol{w}$$

The Hessian $H = \nabla^2 P(\boldsymbol{w})$ has the positive eigenvalues $(21 \pm \sqrt{421})/10$, and thus $P(\boldsymbol{w})$ strictly convex in $\mathbb{R}^2$. The only critical point for $P(\boldsymbol{w})$ is given by

$$\nabla P(\boldsymbol{w}) = H\boldsymbol{w} = \boldsymbol{0} \quad \text{oder} \quad \boldsymbol{w} = \boldsymbol{w}^\star = \boldsymbol{0}$$

where $P(\boldsymbol{w})$ has a global minimum. Furthermore it holds

$$\nabla P(\boldsymbol{w}) \cdot A\boldsymbol{w} = \boldsymbol{w}^\top H^\top A\boldsymbol{w} = \boldsymbol{w}^\top[-A^\top S^{-\top}S^{-1}]A\boldsymbol{w} = -\|S^{-1}A\boldsymbol{w}\|^2 < 0$$

and therefore $P(\boldsymbol{w})$ is a strict Lyapunov Funktion for the system $\boldsymbol{w}'(t) = A\boldsymbol{w}$. According to the stability theorem presented in the lecture, $\boldsymbol{w}^\star$ is locally asymptotically stable. Moreover it is shown as follows that $\boldsymbol{w}^\star$ is globally asymptotically stable.

As with the calculation from the lecture, it holds that $S^{-\top}S^{-1}A\boldsymbol{w} = -\nabla P(\boldsymbol{w})$, $A\boldsymbol{w} = -SS^\top\nabla P(\boldsymbol{w})$ and

$$D_tP(\boldsymbol{w}(t)) = \nabla P(\boldsymbol{w}(t))^\top\boldsymbol{w}'(t) = -\nabla P(\boldsymbol{w}(t))^\top SS^\top\nabla P(\boldsymbol{w}(t)) = -\|S^\top\nabla P(\boldsymbol{w}(t))\|^2.$$

Therefore $P(\boldsymbol{w}(t))$ is non-increasing, independently of initial conditions $\boldsymbol{w}_0$. It follows

$$\int_0^\infty \|S^\top\nabla P(\boldsymbol{w}(s))\|^2 ds = -\int_0^\infty D_sP(\boldsymbol{w}(s))ds = P(\boldsymbol{w}_0) - \lim_{t\to\infty} P(\boldsymbol{w}(t)) \le P(\boldsymbol{w}_0)$$

i.e., $\|S^\top\nabla P(\boldsymbol{w}(t))\|$ is integrable over $[0,\infty)$, and it must be then that

$$0 = \lim_{t\to\infty}\|S^\top\nabla P(\boldsymbol{w}(t))\|^2 = \lim_{t\to\infty}\|S^\top H\boldsymbol{w}(t)\|^2 = \lim_{t\to\infty}\|S^\top[S^{-\top}S^{-1}A]\boldsymbol{w}(t)\|^2$$

$$= \lim_{t\to\infty}\|S^{-1}A\boldsymbol{w}(t)\|^2 = \lim_{t\to\infty}\|S^{-1}[S\Lambda S^{-1}]\boldsymbol{w}(t)\|^2 = \lim_{t\to\infty}\|\Lambda S^{-1}\boldsymbol{w}(t)\|^2$$

$$= \lim_{t\to\infty} \boldsymbol{w}(t)^\top [S^{-\top}\Lambda^2 S^{-1}]\boldsymbol{w}(t) \geq \lambda_{\min}^2 \lim_{t\to\infty} \|\boldsymbol{w}(t)\|^2$$

where the last inequality follows since the matrix $[S^{-\top}\Lambda^2 S^{-1}]$ is SPD, namely with eigenvalues

$$[(3-\sqrt{5})/2]^2 = \lambda_{\min}^2 < \lambda_{\max}^2 = [(3+\sqrt{5})/2]^2.$$

Finally it follows that

$$0 = \lim_{t\to\infty} \|\boldsymbol{w}(t)\| = \lim_{t\to\infty} \|\boldsymbol{w}(t) - \boldsymbol{w}^\star\|$$

independently of the initial conditions $\boldsymbol{w}_0$, and thus the equilibrium $\boldsymbol{w}^\star$ is globally asymptotically stable.

An apparently more comfortable method to reveal the stability might run as follows, but it does not work. One multiplies the differential equation $\boldsymbol{w}'(t) = A\boldsymbol{w}(t)$ with $\boldsymbol{w}(t)$, in order integrate the result advantageously over time,

$$-\frac{1}{2}D_t\|\boldsymbol{w}(t)\|^2 = \boldsymbol{w}(t)^\top A\boldsymbol{w}(t) = \frac{1}{2}\boldsymbol{w}(t)^\top (A + A^\top)\boldsymbol{w}(t).$$

Although the real parts of the eigenvalues $\{\lambda_{\min}, \lambda_{\max}\}$ of the matrix A are always negative, the eigenvalues of the matrix

$$B = A + A^\top = \frac{1}{m}\left[\begin{array}{cc} 0 & 1-k \\ 1-k & -2c \end{array}\right]$$

are given by $[-c \pm \sqrt{c^2 + (k-1)^2}]/m$, and one of these is always non-negative.

• Exercise 9: Predator Prey Model

◦ Task

Show that the function

$$F(x, y) = -a_2 \ln(x) + b_2 x - a_1 \ln(y) + b_1 y$$

is a Lyapunov function for the predator-prey model,

$$x' = (a_1 - b_1 y)x, \quad y' = (b_2 x - a_2)y, \quad a_1, a_2, b_1, b_2 > 0.$$

◦ Solution

Let $(x(t), y(t))$ be a solution curve. It follows

$$D_t P(x(t), y(t)) = a_2 x'(t)/x(t) - b_2 x'(t) + a_1 y'(t)/y(t) - b_1 y'(t)$$

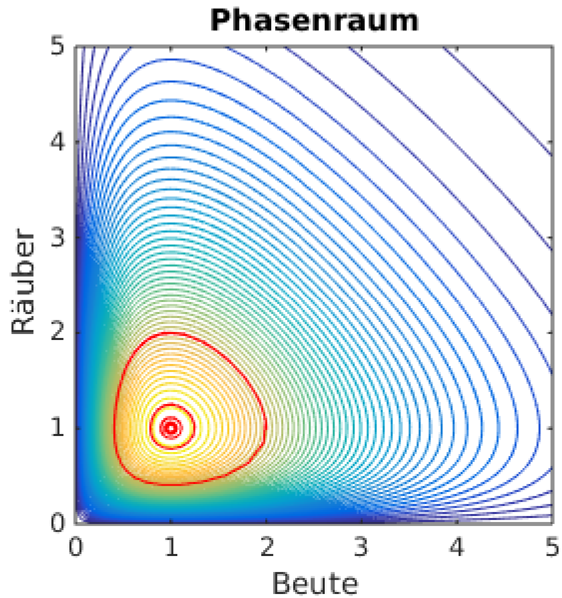
$$= (a_1 - b_1 y(t))(a_2 - b_2 x(t)) + (b_2 x(t) - a_2)(a_1 - b_1 y(t)) = 0$$

and therefore P remains constant in a solution curve, i.e., the solution curves lie in a niveau-curve of the function P. A Matlab code for the graphical representation of the niveau-curves of P is given as follows.

```
% setup figure
    h1 = figure(1); close(h1); h1 = figure(1);
    set(h1,'Position',[10 10 300 300]);
```

```
% Parameter
    a1 = 1; b1 = 1; a2 = 1; b2 = 1;

% Rauber-Beute Modell
    rb = @(t,X) [(a1 - b1*X(2))*X(1);(b2*X(1) - a2)*X(2)];

% Gitter fuer Niveau Kurven im Phasenraum
    n = 101;
    xmin = 0; xmax = 5; ymin = 0; ymax = 5;
    x = linspace(xmin,xmax,n)';
    y = linspace(ymin,ymax,n);
    xx = kron(x,ones(size(y)));
    yy = kron(ones(size(x)),y);

% Auswertung von P auf dem Gitter
    zz = a2*log(xx) - b2*xx + a1*log(yy) - b1*yy;
% geht um die Skalierung fuer die grafische Darstellung
    zz = exp(zz);

    zmin = min(zz(:));
    zmax = max(zz(:));

% Verteilung von Niveau-Kurven
    m = 50;
    w = linspace(zmin,zmax,m);

% grafische Darstellung der Niveau-Kurven
    contour(xx,yy,zz,w);
    axis([xmin xmax ymin ymax])
    pbaspect([1 1 1]);
    xlabel('Beute')
    ylabel('Ruber');
    title('Phasenraum');

% Untersuchung der Stabilitaet des Gleichgewichts
    hold on;
    for k=1:5
        X0 = [a2/b2;a1/b1]+[1/k^2;0];
        tspan = [0,10];
        [t,X] = ode15s(rb,tspan,X0);
        plot(X(:,1),X(:,2),'r')
    end
    hold off;
```

The result is given as follows.



Since the red curves remain close to the equilibrium, then according to these calculations the equilibrium is stable. To show this stability theoretically it will be shown that $F(x, y) = -P(x, y)$ is a Lyapunov function. First with $\boldsymbol{x} = (x, y)$ and $\boldsymbol{f}(\boldsymbol{x}) = ((a_1 - b_1 y)x, (b_2 x - a_2)y)$ the predator prey system can be rewritten as $\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t))$. It holds

$$\nabla F(x, y) = (-a_2/x + b_2, -a_1/y + b_1), \quad \nabla^2 F(x, y) = \begin{bmatrix} a_2/x^2 & 0 \\ 0 & a_1/y^2 \end{bmatrix}$$

$$\nabla F(x, y) \cdot \boldsymbol{f}(x, y) = (a_1 - b_1 y)(-a_2 + b_2 x)x/x + (b_2 x - a_2)(-a_1 + b_1 y)y/y = 0$$

$$\nabla F(a_2/b_2, a_1/b_1) = \mathbf{0}, \quad \nabla^2 F(a_2/b_2, a_1/b_1) = \begin{bmatrix} b_2^2/a_2 & 0 \\ 0 & b_1^2/a_1 \end{bmatrix}$$

Therefore F has a local minimum in $(a_2/b_2, a_1/b_1)$ and $\nabla F(x, y) \cdot \boldsymbol{f}(x, y) = 0$ holds. It follows that $(a_2/b_2, a_1/b_1)$ is a stable equilibrium.

• Exercise 10: Stochastic Growth, Waiting Lines

○ Task

Let $p_n(t) = P\{X(t) = n\}$ be the probability that n customers are waiting to be served at time t, and there are two cashiers. The system of ODEs for these probabilities is:

$$\begin{aligned} p_0'(t) &= -b_0 p_0(t) + d_1 p_1(t) \\ p_n'(t) &= b_{n-1} p_{n-1}(t) - (b_n + d_n) p_n(t) + d_{n+1} p_{n+1}(t), \quad 1 \le n \le N-1 \\ p_N'(t) &= b_{N-1} p_{N-1}(t) - d_N p_N(t) \end{aligned}$$

where the coefficients $\{b_n\}$ and $\{d_n\}$ are determined as follows.

- The average time between customer arrivals is $c = 1/b_n$ and is independent of the number of cashiers.

- If there is only one customer, then $s = 1/d_1$ is the average service time when only one cashier is in operation.
- When there are at least two customers, then $s/2 = 1/d_n$, $2 \le n \le N$, is the average service time when two cashiers are in operation.

Therefore it holds

$$b_n = 1/c, \quad d_n = \begin{cases} 2/s, & 2 \le n \le N \\ 1/s, & n = 1 \end{cases}$$

Let $\{p_n^\star\}$ be the stationary state for $X(t)$. Show with $\rho = s/c$,

$$E[X^\star] = p_0 \rho \frac{N(\rho/2)^{N+1} - (N+1)(\rho/2)^N + 1}{(1-\rho/2)^2}, \quad p_0 = \frac{1-\rho/2}{1+\rho/2-\rho(\rho/2)^N}$$

and

$$E[X^\star] \overset{\rho\to 2}{\longrightarrow} \frac{N(N+1)}{1+2N}, \quad E[X^\star] \overset{N\to\infty}{\longrightarrow} \frac{4\rho}{4-\rho^2} \equiv L_2(\rho) \quad \text{für } \rho \in (0,2)$$

◦ Solution

For the system of ODEs $\boldsymbol{p}' = A\boldsymbol{p}$ the matrix A is given by

$$A = \begin{bmatrix} -\frac{1}{c} & \frac{1}{s} & & & & 0 \\ \frac{1}{c} & -\frac{1}{c}-\frac{1}{s} & \frac{2}{s} & & & \\ & \frac{1}{c} & -\frac{1}{c}-\frac{2}{s} & \frac{2}{s} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{c} & -\frac{1}{c}-\frac{2}{s} & \frac{2}{s} \\ 0 & & & & \frac{1}{c} & -\frac{2}{s} \end{bmatrix}$$

For the stationary state it follows from the first equation of the system $A\boldsymbol{p}^\star = \boldsymbol{0}$ that

$$0 = -\frac{p_0^\star}{c} + \frac{p_1^\star}{s}, \quad p_1^\star = \rho p_0^\star, \quad \rho = \frac{s}{c}$$

and from the second equation

$$0 = \left(\frac{p_0^\star}{c} - \frac{p_1^\star}{s}\right) - \frac{p_1^\star}{c} + \frac{2p_2^\star}{s}, \quad p_2^\star = \frac{\rho}{2} p_1^\star = \frac{\rho^2}{2} p_0^\star$$

etc.,

$$p_n^\star = \frac{\rho^n}{2^{n-1}} p_0^\star, \quad 1 \le n \le N$$

The remaining probability $p_0^\star$ is given by

$$1 = \sum_{n=0}^{N} p_n^\star, \quad p_0^\star = 1/\left[1 + \sum_{n=1}^{N} \frac{\rho^n}{2^{n-1}}\right] = 1/\left[1 + 2\sum_{n=1}^{N} \left(\frac{\rho}{2}\right)^n\right] = 1/\left[1 + 2\left(-1 + \frac{1-(\rho/2)^{N+1}}{1-\rho/2}\right)\right]$$

or

$$\rho_0^\star = \frac{1-\rho/2}{1+\rho/2-\rho(\rho/2)^N}$$

The expected value of the length of the waiting line (for the 2 cashiers) ist

$$\mathbb{E}[X^\star] = \sum_{n=0}^{N} n p_n^\star = p_0^\star \sum_{n=0}^{N} n \frac{\rho^n}{2^{n-1}} = \rho p_0^\star \sum_{n=0}^{N} n \left(\frac{\rho}{2}\right)^{n-1} = \rho p_0^\star \, \frac{d}{dz} \sum_{n=0}^{N} z^n \Bigg|_{z=\rho/2}$$

$$= \rho p_0^\star \, \frac{d}{dz} \left[\frac{1-z^{N+1}}{1-z}\right]\Bigg|_{z=\rho/2} = \rho p_0^\star \frac{1-(N+1)(\rho/2)^N + N(\rho/2)^{N+1}}{(1-(\rho/2))^2}$$

$$= \rho \frac{1-\rho/2}{1+\rho/2-\rho(\rho/2)^N} \frac{1-(N+1)(\rho/2)^N + N(\rho/2)^{N+1}}{(1-(\rho/2))^2} = \frac{\rho}{1-\rho/2} \frac{1-(\rho/2)^N - N(\rho/2)^N(1-\rho/2)}{1+\rho/2-\rho(\rho/2)^N}$$

and furthermore with $1-(\rho/2)^N = (1-\rho/2)\sum_{n=0}^{N-1}(\rho/2)^n$,

$$\mathbb{E}[X^\star] = \rho \frac{\sum_{n=0}^{N-1}(\rho/2)^n - N(\rho/2)^N}{1+\rho/2-2(\rho/2)^{N+1}}.$$

With $\sigma = \rho/2$ and L'Hôpital's rule,

$$\lim_{\rho\to 2} \mathbb{E}(X^\star) = \lim_{\rho\to 2} \rho \cdot \lim_{\sigma\to 1} \frac{\sum_{n=0}^{N-1} \sigma^n - N\sigma^N}{1+\sigma-2\sigma^{N+1}} = 2 \lim_{\sigma\to 1} \frac{\sum_{n=0}^{N-1} n\sigma^{n-1} - N^2\sigma^{N-1}}{1-2(N+1)\sigma^N} = \frac{N(N-1)-2N^2}{1-2(N+1)} = \frac{N^2+N}{2N+1}$$

Finally,

$$\lim_{N\to\infty} \mathbb{E}(X^*) = \lim_{N\to\infty} \mathbb{E}(X^*) \frac{\rho}{1-\rho/2} \frac{1-(\rho/2)^N - N(\rho/2)^N(1-\rho/2)}{1+\rho/2-\rho(\rho/2)^N} = \frac{\rho}{1-\rho/2} \cdot \frac{1-0-0}{1+\rho/2-0}$$

or

$$L_2(\rho) = \frac{4\rho}{4-\rho^2}.$$

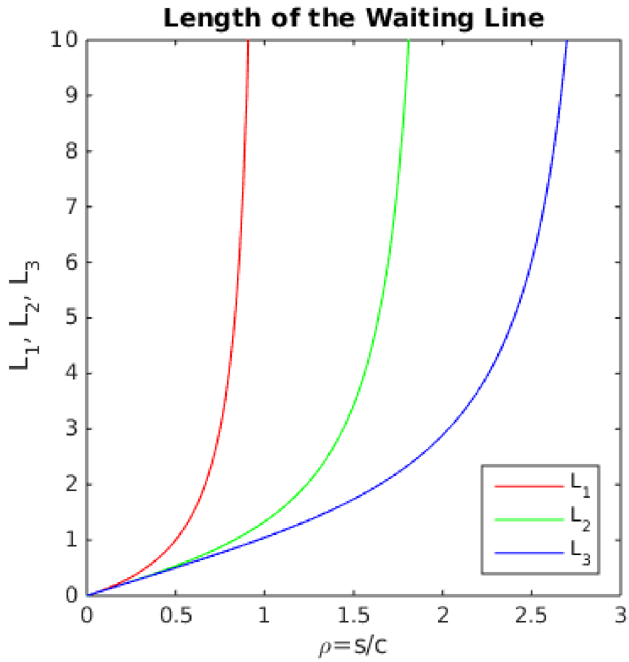
Similarly one obtains

$$L_3(\rho) = \frac{\rho(18+6\rho-\rho^2)}{(3-\rho)(6+4\rho+\rho^2)}$$

and for k cashiers,

$$L_k(\rho) = \rho + \left[\frac{k}{\rho} - 1 + \frac{(k-1)!}{\rho^{k+1}}(\rho-k)^2 \sum_{n=0}^{k-1} \frac{\rho^n}{n!}\right]^{-1}.$$

If desired that the length of the waiting line does not exceed 5 customers, then the manager can decide on the basis of the quotient $\rho = s/c$ how many cashiers should be opened.



• Exercise 11: Stochastic Transitions between Building Floors

◦ Task

Let $i \in \{0(\mathrm{G}), 1, \ldots, N\}$ be an index for the floor of a building. Let X be a random variable which satisfies $X(n) = i$ if an elevator is at the ith floor after n time steps each of duration Δt. For $p_i(n) = P(X(n) = i)$ and $\boldsymbol{p}(n) = \{p_i(n)\}_{i=0}^N$ let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be a stochastic matrix, where $\boldsymbol{p}(n) = P^\top \boldsymbol{p}(n-1)$ holds.

It is assumed that Δt is so small that jumps of two or more floors in a time interval of length Δt are not possible. Otherwise all transitions with neighboring floors is possible, also that there be no transition. Thus, p is genuinely tridiagonal but otherwise an arbitrary stochastic matrix. In particular it is not necessarily the case that P is symmetric.

For an $N \in \mathbb{N}$ choose such a stochastic matrix P and carry out the following calculations. Find a stationary state (equilibrium) of the states $\boldsymbol{p}^\star$ of the elevator. Confirm that $\boldsymbol{p}_0^\top P^n \to \boldsymbol{p}^{\star\top}$, $n \to \infty$, holds for an arbitrary initial distribution $\boldsymbol{p}_0$. Confirm further that $P^n > 0$ holds for an $n \in \mathbb{N}$, and that all rows of P^n converge to the state $\boldsymbol{p}^{\star\top}$ for ever increasing n. How can the theorem from the lecture about such chains be applied here? Under which conditions are all the entries of $\boldsymbol{p}^\star$ equal?

With the same P and $\boldsymbol{p}^\star$ make the following random walk. Choose an arbitrary floor initially with index i. Determine a next floor j according to the transition probability $P_{i,j}$. This can be done, e.g., with a uniformly distributed random variable z in $[0, 1]$: $j = i - 1$ if $z \in [0, P_{i,i-1}]$, $j = i$ if $z \in (P_{i,i-1}, P_{i,i-1} + P_{i,1}]$ and $j = i + 1$ if $z \in (P_{i,i-1} + P_{i,1}, 1]$. Then overwrite i with j and carry out such steps several times until the relative frequency distribution for the floors becomes stable. Compare this relative frequency distribution with $\boldsymbol{p}^*$.

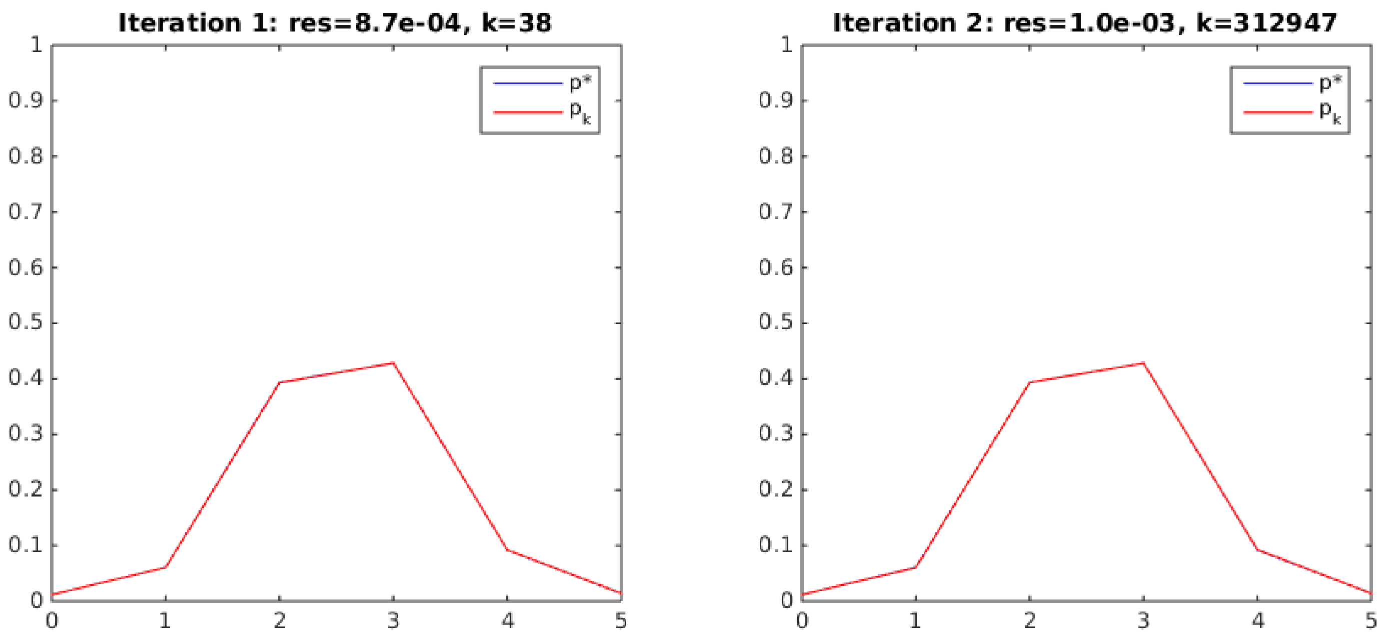
◦ Solution

The simulation is carried out in the following Matlab code.

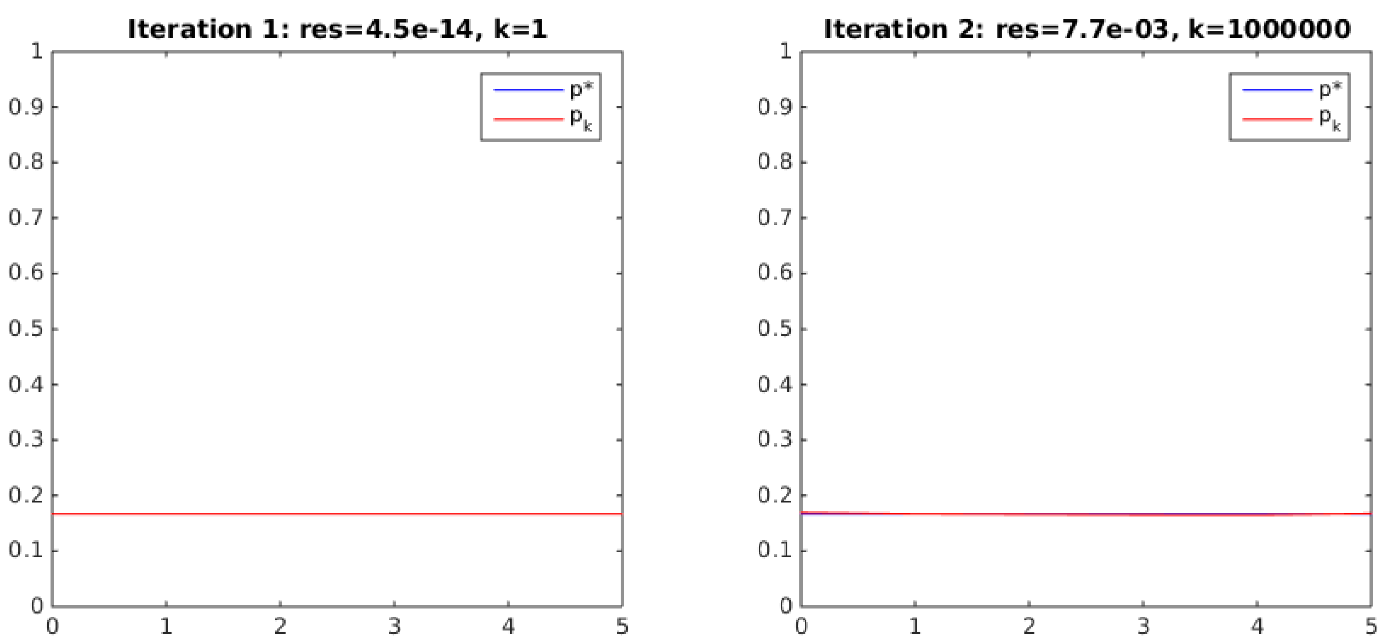
```
h1 = figure(1); close(h1); h1 = figure(1);
set(h1,'Position',[10 10 1000 400]); % setup figure

kmax = 1.0e6;                        % max number of iterations
tol = 1.0e-3;                        % convergence criterion

N  = 5;                              % N+1 floors

example = 1;
switch example
    case 1
        P  = rand(N+1,3);
        P  = spdiags(P,[-1 0 1],N+1,N+1);
        Ps = P*ones(N+1,1);          % stochastic matrix P
        P  = diag(1./Ps)*P;          % is not symmetric
    case 2
        P = zeros(N+1,N+1);
        P(1,1) = rand(1);
        P(1,2) = 1-P(1,1);
        for i=2:N
            P(i,i-1) = P(i-1,i);
            P(i,i)   = (1-P(i,i-1))*rand(1);
            P(i,i+1) = 1-P(i,i-1)-P(i,i);
        end
        P(N+1,N) = P(N,N+1);         % stochastic matrix P
        P(N+1,N+1) = 1-P(N+1,N);     % is symmetric
end

[V,D] = eig(full(P'));
i = find(abs(diag(D)-1) < tol);
p = V(:,i);                          % eigenvector with eigenvalue = 1
p = p/sum(p);                        % P'*p = p, 1'*p = 1

subplot(1,2,1)
Pk = speye(N+1);
for k=1:kmax
    Pk = Pk*P;                       % P^k
    pk = Pk'*ones(N+1,1)/(N+1);      % column average of P^k
    if (norm(p-pk) < tol*norm(p))
        break;
    end
end
if (k == kmax)
    warning(sprintf('Iteration 1: keine Konvergenz mit kmax=%0.0f',kmax))
end
```

```
plot(0:N,p,'b',0:N,pk,'r')
axis([0 N 0 1])
legend('p*','p_k')
title(sprintf('Iteration 1: res=%0.1e, k=%0.0f', ...
    norm(p-pk)/norm(p),k))
drawnow;

Ps = [[0;diag(P,-1)],diag(P),[diag(P,+1);0]];
Ps = [Ps(:,1), ...
      Ps(:,1)+Ps(:,2), ...
      Ps(:,1)+Ps(:,2)+Ps(:,3)];      % diagonalwise row sum of P
pk = zeros(N+1,1);
j  = randi(N+1);                     % initial floor

subplot(1,2,2)
for k=1:kmax
    i  = j;                          % update current floor
    pk(i) = pk(i) + 1;               % ith floor has been visited
    if (norm(p-pk/k) < tol*norm(p))
        break;                       % stop if average number of visits
    end                              % for all floors agrees with p
    z = rand(1);
    j = (i-1)* (z <= Ps(i,1)) ...    % next floor
      +     i*((z <= Ps(i,2)) && (Ps(i,1) < z)) ...
      + (i+1)*((z <= Ps(i,3)) && (Ps(i,2) < z));
end
if (k == kmax)
    warning(sprintf('Iteration 2: keine Konvergenz mit kmax=%0.0f',kmax))
end
pk = pk/k;
plot(0:N,p,'b',0:N,pk,'r')
axis([0 N 0 1])
legend('p*','p_k')
title(sprintf('Iteration 2: res=%0.1e, k=%0.0f',norm(p-pk)/norm(p),k))
drawnow;
```

The results for $N = 5$ are shown graphically as follows, first for the case that P is symmetric,



and then for the case that P is not symmetric,



One notices that the number of iterations necessary to reach for a stable frequency distribution is much larger than the number required for the sequence of matrix powers to converge.

Let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be tridiagonal with $P_{i,j} \neq 0$, $|i-j| \leq 1$. With

$$\mu_{i,j} = \min\{i,j\}, \quad \nu_{i,j} = \max\{i,j\}.$$

it follows

$$P_{i,j} \begin{cases} \neq 0, & \nu_{i,j} - \mu_{i,j} \leq 1 \\ = 0, & \text{otherwise} \end{cases}$$

and

$$(P^2)_{i,j} = \sum_{k=1}^{N+1} p_{i,k}p_{k,j} = \sum_{k=i-1}^{i+1} p_{i,k}p_{k,j} = \sum_{k=j-1}^{j+1} p_{i,k}p_{k,j} = \sum_{k=\nu_{i,j}-1}^{\mu_{i,j}+1} p_{i,k}p_{k,j} \begin{cases} \neq 0, & \nu_{i,j} - \mu_{i,j} \leq 2 \\ = 0, & \text{sonst} \end{cases}$$

Thus P has the bandwidth 1, while P^2 has the bandwidth 2. Similarly P^k has the bandwidth k, $1 \leq k \leq N$, and P^k is full for $k \geq N+1$. The theorem from the lecture can be applied for P to argue the existence of an equilibrium theoretically.

For a stochastic matrix P there holds

$$P\mathbf{1} = \mathbf{1}$$

In case P is symmetric, it follows

$$P^\top \mathbf{1} = P\mathbf{1} = \mathbf{1}$$

and therefore

$$\boldsymbol{p}^\star = \mathbf{1}/N$$

is an equilibrium.

To determine a condition for the condition that the entries of $\boldsymbol{p}^\star$ are equal, let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be written in the form,

$$P = \begin{bmatrix} 1-\alpha_0 & \alpha_0 & & & 0 \\ \beta_1 & 1-\alpha_1-\beta_1 & \alpha_1 & & \\ & \ddots & \ddots & & \\ & & \beta_{N-1} & 1-\alpha_{N-1}-\beta_{N-1} & \alpha_{N-1} \\ 0 & & & \beta_N & 1-\beta_N \end{bmatrix}$$

with

$$0 < \alpha_i < 1, \quad 0 < \beta_i < 1, \quad 1 \leq i+1, j \leq N.$$

To finde the equilibrium, it must be solved for $\boldsymbol{p}^\star = \{p_i^\star\}_{i=0}^N$ in

$$P^\top \boldsymbol{p}^\star = \boldsymbol{p}^\star, \quad \sum_{i=0}^N p_i^\star = 1, \quad p_i^\star \geq 0$$

Through sucessive addition of the rows of $(P^\top - I)$ the system $(P^\top - I)\boldsymbol{p}^\star$ takes the form

$$\begin{bmatrix} -\alpha_0 & \beta_1 & & & 0 \\ 0 & -\alpha_1 & \beta_2 & & \\ & \ddots & \ddots & \dots & \\ & & 0 & -\alpha_{N-1} & \beta_N \\ 0 & & & -\alpha_{N-1} & \beta_N \end{bmatrix} \boldsymbol{p}^\star = 0.$$

The result is the solution,

$$p_i^\star = p_0^\star \Big[\prod_{j=0}^{i-1} \alpha_j \Big] / \Big[\prod_{j=1}^{i} \beta_j \Big], \quad \frac{1}{p_0^\star} = 1 + \Big[\prod_{j=0}^{i-1} \alpha_j \Big] / \Big[\prod_{j=1}^{i} \beta_j \Big].$$

If all entries of $\boldsymbol{p}^\star$ are equal, then it must hold that

$$p_1^\star = p_0^\star \alpha_0/\beta_1 \quad \Rightarrow \quad \alpha_0 = \beta_1$$

$$p_2^\star = p_1^\star \alpha_1/\beta_2 \quad \Rightarrow \quad \alpha_1 = \beta_2$$

etc.

$$\alpha_i = \beta_{i+1}, \quad i = 0, \dots, N-1$$

and so P must be symmetric.

• Exercise 12: Infection Model

◦ Task

For the parameters $\beta = 100$, $\mu = 0.001$, $\gamma = 0.4$ and $\lambda = 5 \cdot 10^{-6}$ implement the SIR model,

$$S' = \beta - (\mu + \lambda I)S, \quad I' = (\lambda S - \mu - \gamma)I, \quad R' = -\mu R + \gamma I$$

and plot the results in time and in phase space. Prove that the equilibrium obtained is locally asymptotically stable.

◦ Solution

The initial value problem is solved with the following Matlab code.

```
% model parameters
    be = 100;
    mu = 0.001;
    ga = 0.4;
    la = 5*10^(-6);

% SIR model
    f = @(t,y) [be-(mu+la*y(2))*y(1);
              (la*y(1)-mu-ga)*y(2);
              -mu*y(3)+ga*y(2);];

% final time
    T = 13*365;

% initial values
    y0 = [10^5 100 0]';

% compute solution
[t,y]=ode45(f,[0 T],y0);
```

```
% graphical representation
    h1 = figure(1); close(h1); h1 = figure(1);
    set(h1,'Position',[20 20 500 500]);
    h2 = figure(2); close(h2); h2 = figure(2);
    set(h2,'Position',[20 20 1500 500]);

% phase space
    figure(1)

    plot3(y(:,1),y(:,2),y(:,3),'LineWidth',3);
    axis([60000 100000 0 2300 0 40000])
    grid on;
    view(35,50);
    xlabel('S(t)');
    ylabel('I(t)');
    zlabel('R(t)');
    title('Phase Space')

% dynamic
    figure(2)

    subplot (3,1,1)
    plot(t,y(:,1))
    ylabel('S(t)');
    text(5100,15000,'t');
    axis([0 5000 0 100000])
    title('susceptible')

    subplot(3,1,2)
    plot(t,y(:,2))
    text(5100,450,'t');
    ylabel('I(t)');
    axis([0 5000 0 3000])
    title('infected')

    subplot(3,1,3)
    plot(t,y(:,3))
    text(5100,5500,'t');
    ylabel('R(t)');
    axis([0 5000 0 40000])
    title('recovered')
```

The results for an apparently stable endemic equilibrium are shown graphically as follows.