Mathematical Modelling in the Natural Sciences
SS20, Exercises, Sheet 8

Solutions to be presented on 17. June 2020

It is recommended to use an explicit upwind differencing approach for numerical solutions to these problems, which can be described as follows. To solve a scalar conservation law $u_t + f(u)_x = 0$ on a space-time grid $\{(x_i, t^n) : x_i = i\Delta x, t^n = n\Delta t, i \in \mathbb{Z}, n \in \mathbb{N}_0\}$ the explicit upwind scheme is given by

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x} = 0$$

where $U_i^k \approx u(x_i, t^n)$ are cell-centered approximations to the solution and the *numerical flux function* F is an approximation to the flux on cell interfaces given by

$$F^n_{i+\frac{1}{2}} = \frac{f(U_i) + f(U_{i+1})}{2} - |a_{i+\frac{1}{2}}| \frac{U_{i+1} - U_i}{2}, \quad a_{i+\frac{1}{2}} = \frac{f(U_{i+1}) - f(U_i)}{U_{i+1} - U_i}.$$

Note for $f(u) = au$,

$$F_{i+\frac{1}{2}} = \frac{a}{2}[U_i + U_{i+1}] - \frac{|a|}{2}[U_{i+1} - U_i]$$

that if $a > 0$ holds, then backward differencing is used for the term $f(u)_x$

$$F_{i+\frac{1}{2}} = aU_i, \quad \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x} = a\frac{U_i - U_{i-1}}{\Delta x} = a\frac{U_{i+1} - U_{i-1}}{2\Delta x} - \frac{a\Delta x}{2}\frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2}$$

and if $a < 0$ holds, then forward differencing is used,

$$F_{i+\frac{1}{2}} = aU_{i+1}, \quad \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x} = a\frac{U_{i+1} - U_i}{\Delta x} = a\frac{U_{i+1} - U_{i-1}}{2\Delta x} + \frac{a\Delta x}{2}\frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2}.$$

In both cases, the viscous counterpart $u_t + f(u)_x = \nu u_{xx}$ of the scalar conservation law can be seen with numerical artificial viscosity $\nu = |a|\Delta x/2$. For explicit programming details, see this code for an implementation of the approach for Burger's equation.

1. Let traffic velocity be related to density according to $u(\rho) = u_{\max}(1 - \rho_{\max})$ for $u_{\max}, \rho_{\max} > 0$. The traffic density is then given by $f(\rho) = \rho u(\rho)$. Write a Matlab code to perform the following calculations to model the reaction of traffic to a narrowing road.
   (a) Solve the Riemann Problem with $\rho_{\mathrm{l}} = \rho_{\max}/4$ and $\rho_{\mathrm{r}} = 3\rho_{\max}/4$.
   (b) Sketch the trajectories and the characteristics.
   (c) Sketch $\rho(x, t)$ and $u(x, t)$ for a fixed times $t > 0$.
   (d) Determine the vehicle velocity $v(t)$ along fixed trajectories.

2. Repeat the last exercise with $\rho_{\mathrm{l}} = 3\rho_{\max}/4$ and $\rho_{\mathrm{r}} = \rho_{\max}/4$ to model the reaction of traffic to a widening road. How would the velocity model have to be changed so that autos further back from the widening would not have to drive initially more slowly than those autos nearer to the widening?

3. Let traffic velocity be related to density according to $u(\rho) = u_{\max}(1 - \rho_{\max})$ for $u_{\max}, \rho_{\max} > 0$. The traffic flux is then given by $f(\rho) = \rho u(\rho)$. Write a Matlab code implementing an explicit upwinding approach to solve the initial value problem,

$$\rho_t + f(\rho)_x = 0, \quad \rho(x,0) = \hat{\rho} + \sigma(x)$$

where σ is a local perturbation chosen so that its resulting wave optionally

(a) travels in the opposite direction as traffic,

(b) stands still or

(c) travels in the same direction as traffic.

4. Let traffic velocity be related to density according to the model of night-time driving,

$$u(\rho) = \begin{cases} U_0 & \rho < \rho_{\mathrm{a}} \\ c\rho & \rho_{\mathrm{a}} \le \rho \le \rho_{\mathrm{b}} \\ U_1(\rho_{\max} - \rho) & \rho > \rho_{\mathrm{b}}. \end{cases}$$

Let $U_0 = 1$, $\rho_{\mathrm{a}} = \frac{1}{10}$, $\rho_{\mathrm{b}} = \frac{3}{10}$, $c = 10$, $U_1 = \frac{30}{7}$ and $\rho_{\max} = 1$. As usual, the traffic flux is given by $f(\rho) = \rho u(\rho)$. Let $\hat{\rho}$ be the initial density of a column of N autos. Taking a Lagrangian perspective let auto trajectories be determined by a solution to the initial-value problem

$$\begin{cases} x_k'(t) = u(\rho_k(t)), \quad k = 1, \ldots, N-1 \\ x_N'(t) = u(0) \end{cases} \qquad \frac{1}{x_{k+1}(0) - x_k(0)} = \hat{\rho}\left(\frac{x_{k+1}(0) + x_k(0)}{2}\right)$$

where $\rho_k(t)$ is chosen on the one hand to be a density function $\rho_k^{(\mathrm{n})}$ designed to obtain a realistic solution to the night-time driving problem,

$$\rho_k^{(\mathrm{n})}(t) = \frac{1}{x_{k+1}(t) - x_k(t)}$$

and on the other hand chosen to be a density function $\rho_k^{(\mathrm{e})}$ be designed to obtain an entropy solution to the night-time driving problem,

$$\rho_k^{(\mathrm{e})}(t) = \frac{1}{2}\left\{\frac{1}{x_{k+1}(t) - x_k(t)} + \frac{1}{x_k(t) - x_{k-1}(t)}\right\}.$$

(a) Compare the results for the two density functions by solving the Riemann problem with each using $\rho_{\mathrm{l}} = 1$ and $\rho_{\mathrm{r}} = 0$.

(b) Using $\rho^{(\mathrm{n})}$ demonstrate instability and clustering by starting optionally with the constant initial densities $\hat{\rho} = \rho_{\mathrm{a}}$, $(\rho_a + \rho_b)/2$ and ρ_{b}.

5. Repeat the last exercise by taking an Eulerian perspective to compute densities according to an upwind differencing approach by solving the scalar conservation law $\rho_t + f(\rho)_x = 0$ with $f(\rho) = \rho u(\rho)$, where $u(\rho)$ is given by the night-time velocity model. Note that with the upwind scheme the entropy solution will be computed. In order to obtain the more realistic solution seek a numerical flux function which is stable but has more of a focus on the upstream state.

6. Let traffic velocity be related to density according to $u(\rho) = u_{\max}(1 - \rho_{\max})$ for $u_{\max}, \rho_{\max} > 0$. The traffic flux is then given by $f(\rho) = \rho u(\rho)$. In addition to the moving column of autos there are standing autos waiting for a spot in the column. A waiting auto enters only when the

traveling density ρ reaches the threshold $\rho > \rho_{\rm i}$. Let ω be the density of standing autos. The interaction between the two types of autos is modelled by the initial value problem,

$$\omega_t = -\alpha(\rho)\omega, \quad \rho_t + f(\rho)_x = \alpha(\rho)\omega, \quad \alpha(\rho) = \hat{\alpha}(\rho > \rho_{\rm i}), \quad \rho(x,0) = \rho_0(x), \quad \omega(x,0) = \omega_0(x)$$

where ω_0 and ρ_0 are the respective initial densities. Write a Matlab code implementing an explicit upwinding approach to solve the initial value problem given initial conditions

$$\omega_0(x) = \begin{cases} \omega_{\rm l}, & x < 0 \\ \omega_{\rm r}, & x > 0 \end{cases} \qquad \rho_0(x) = \begin{cases} \rho_{\rm l}, & x < 0 \\ \rho_{\rm r}, & x > 0 \end{cases}$$

where, for instance, $\omega_{\rm r} = 0 < \omega_{\rm l}$ and $\rho_{\rm l} < \rho_{\rm i} < \rho_{\rm r}$. With $\rho_{\rm l} < \rho_{\rm r}$ there emerges a traffic jam. With $\rho_{\rm r} > \rho_{\rm i}$ the density of the slowed autos in the region $x > 0$ exceeds the threshold to bring the autos waiting with density $\omega_{\rm l}$ into the column.