Mathematical Modelling in the Natural Sciences
SS19, Exercises, Sheet 9

Solutions to be presented on 17. June 2019

1. Show that the function $u(x,t) = u_0(x - at)$ satisfies the weak formulation of the convection equation
$$u_t + au_x = 0, \quad u(x,0) = u_0(x).$$

2. Given the solution
$$v^\epsilon(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-y)^2}{4\epsilon t}} u_0(y) dy$$
to the heat equation
$$v_t^\epsilon = \epsilon v_{xx}^\epsilon, \quad v^\epsilon(x,0) = u_0(x)$$
show that the regularized equation
$$u_t^\epsilon + au_x^\epsilon = \epsilon u_{xx}^\epsilon, \quad u^\epsilon(x,0) = u_0(x)$$
is solved with $u^\epsilon(x,t) = v^\epsilon(x - at, t)$ which satisfies $u^\epsilon(x,t) \to u_0(x - at)$ for $\epsilon \to 0$. Assume for simplicity that u_0 is continuous and bounded.

3. Show that for $u_\mathrm{l} > u_\mathrm{r}$ the Riemann problem for Burger's equation,
$$u_t + uu_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} u_\mathrm{l}, & x < 0 \\ u_\mathrm{r}, & x > 0 \end{cases}$$
is solved weakly by the shock $u(x,t) = u_0(x - st)$ where $s = (u_\mathrm{l} + u_\mathrm{r})/2$ is the shock velocity.

4. Define
$$u_0(x) = \begin{cases} u_\mathrm{l}, & x < 0 \\ (u_\mathrm{l} + u_\mathrm{r})/2, & x = 0 \\ u_\mathrm{R}, & x > 0 \end{cases}$$
and
$$w^\epsilon(x) = u_\mathrm{r} + \frac{1}{2}(u_\mathrm{l} - u_\mathrm{r})[1 - \tanh((u_\mathrm{l} - u_\mathrm{r})x/(4\epsilon))]$$
which satisfies
$$w^\epsilon(x) \begin{cases} \to u_\mathrm{l}, & x \to -\infty \\ \to u_\mathrm{r}, & x \to +\infty. \end{cases}$$
Show that
$$u_t + uu_x = \epsilon u_{xx}, \quad u(x,0) = w^\epsilon(x)$$
is solved by $u^\epsilon(x,t) = w^\epsilon(x - st)$, where $s = (u_\mathrm{l} + u_\mathrm{r})/2$. Then show that $u^\epsilon(x,t) \to u_0(x - st)$ for $\epsilon \to 0$.