

Mathematical Modelling in the Natural Sciences

SS19, Exercises, Sheet 4

Solutions to be presented on 1. April 2019

1. Let $p_n(t) = P\{X(t) = n\}$ be the probability that n customers are waiting to be served at time t , and there are two cashiers. The system of ODEs for these probabilities is:

$$\begin{aligned} p'_0(t) &= -b_0 p_0(t) + d_1 p_1(t) \\ p'_n(t) &= b_{n-1} p_{n-1}(t) - (b_n + d_n) p_n(t) + d_{n+1} p_{n+1}(t), \quad 1 \leq n \leq N-1 \\ p'_N(t) &= b_{N-1} p_{N-1}(t) - d_N p_N(t) \end{aligned}$$

where the coefficients $\{b_n\}$ and $\{d_n\}$ are determined as follows.

- The average time between customer arrivals is $c = 1/b_n$ and is independent of the number of cashiers.
- If there is only one customer, then $s = 1/d_1$ is the average service time when only one cashier is in operation.
- When there are at least two customers, then $s/2 = 1/d_n$, $2 \leq n \leq N$, is the average service time when two cashiers are in operation.

Therefore it holds

$$b_n = 1/c, \quad d_n = \begin{cases} 2/s, & 2 \leq n \leq N \\ 1/s, & n = 1 \end{cases}$$

Let $\{p_n^*\}$ be the stationary state for $X(t)$. Show with $\rho = s/c$,

$$E[X^*] = p_0 \rho \frac{N(\rho/2)^{N+1} - (N+1)(\rho/2)^N + 1}{(1 - \rho/2)^2}, \quad p_0 = \frac{1 - \rho/2}{1 + \rho/2 - \rho(\rho/2)^N}$$

and

$$E[X^*] \xrightarrow{\rho \rightarrow 2} \frac{N(N+1)}{1+2N}, \quad E[X^*] \xrightarrow{N \rightarrow \infty} \frac{4\rho}{4-\rho^2} \equiv L_2(\rho) \quad \text{für } \rho \in (0, 2)$$

Solved by Mr. Habring.

2. Define a random variable X where $X(n) = i$ if an elevator is at the i -th floor after n time intervals Δt . Suppose $i \in \{0(\text{G}), 1, \dots, N\}$. For $p_i(n) = P(X(n) = i)$ and $\mathbf{p}(n) = \{p_i(n)\}_{i=0}^N$, let $P \in \mathbb{R}^{(N+1) \times (N+1)}$ be a stochastic matrix containing transition probabilities satisfying $\mathbf{p}(n) = P^\top \mathbf{p}(n-1)$. Assume that Δt is small enough that jumps of two floors with a time interval Δt is impossible, and hence P is tridiagonal but otherwise arbitrary; in particular, P need not be symmetric. For some $N \in \mathbb{N}$ and for sample entries in P , determine the stationary (equilibrium) distribution of states \mathbf{p}^* for the elevator and demonstrate by simulation that $\mathbf{p}_0^\top P^n \rightarrow \mathbf{p}^{*\top}$, $n \rightarrow \infty$, for any starting distribution \mathbf{p}_0 . Show also that $P^n > 0$ for some n , and that all rows of P^n converge to $\mathbf{p}^{*\top}$. Under which circumstances are all entries of \mathbf{p}^* the same? Solved by Mr. Stadler.

3. For the parameters $\beta = 100$, $\mu = 0.001$, $\gamma = 0.4$ and $\lambda = 5 \cdot 10^{-6}$ implement the *SIR* model,

$$S' = \beta - (\mu + \lambda I)S, \quad I' = (\lambda S - \mu - \gamma)I, \quad R' = -\mu R + \gamma I$$

and plot the results in time and in phase space. Prove that the equilibrium obtained is locally asymptotically stable. Solved by Mr. Cardenas.