Mathematical Modelling in the Natural Sciences
SS19, Exercises, Sheet 3

Solutions to be presented on 25. April 2019

1. Show that the equilibrium $(x^\star, y^\star) = (a_2/b_2, a_1/b_1)$ for the Gause equations,

$$x'(t) = (a_1 - b_1 y)x, \quad y'(t) = (a_2 - b_2 x)y, \quad a_1, a_2, b_1, b_2 > 0$$

is unstable and, in particular, a saddle point. Solved by Mr. Fink.

2. Let the damped mass spring model,

$$mu''(t) = f - ku(t) - cu'(t)$$

for $m = 1$, $k = 1$, $f = 1$ and $c = 3$ be written in first order form

$$\boldsymbol{u}'(t) = A\boldsymbol{u}(t) + \boldsymbol{b}, \quad \boldsymbol{u}(t) = (u(t); u'(t)), \quad A = [0, 1; -k, -c]/m, \quad \boldsymbol{b} = (0; f/m).$$

First show that the equilibrium

$$\boldsymbol{u}^\star = (u^\star; 0) = -A^{-1}\boldsymbol{b}$$

is locally asymptotically stable. Then note that the system can also be written as

$$\boldsymbol{w}'(t) = A\boldsymbol{w}(t), \quad \boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^\star.$$

To show that $\boldsymbol{w} = \boldsymbol{0}$ is globally asymptotically stable, derive a function $P(\boldsymbol{w})$ satisfying

$$S^{-\top} S^{-1} A\boldsymbol{w} = -\nabla P(\boldsymbol{w}), \quad AS = S\Lambda, \quad \Lambda = \mathrm{diag}\{\lambda_i\}_{i=1}^2$$

and show that P decreases to its global minimum at $\boldsymbol{w} = \boldsymbol{0}$ along every solution $\boldsymbol{w}(t)$. Solved by Mr. Stadler.

3. Show that the function

$$F(x, y) = -a_2 \ln(x) + b_2 x - a_1 \ln(y) + b_1 y$$

is a Lyapunov function for the predator-prey model,

$$x' = (a_1 - b_1 y)x, \quad y' = (b_2 x - a_2)y, \quad a_1, a_2, b_1, b_2 > 0.$$

Solved by Mr. Habring.