

# Mathematical Modelling in the Natural Sciences

## SS18, Exercises, Sheet 11

*Solutions to be presented on 29. June 2018*

1. Show that the function  $u(x, t) = u_0(x - at)$  satisfies the weak formulation of the convection equation

$$u_t + au_x = 0, \quad u(x, 0) = u_0(x).$$

2. Given the solution

$$v^\epsilon(x, t) = \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4\epsilon t}} u_0(y) dy$$

to the heat equation

$$v_t^\epsilon = \epsilon v_{xx}^\epsilon, \quad v^\epsilon(x, 0) = u_0(x)$$

show that the regularized equation

$$u_t^\epsilon + au_x^\epsilon = \epsilon u_{xx}^\epsilon, \quad u^\epsilon(x, 0) = u_0(x)$$

is solved with  $u^\epsilon(x, t) = v^\epsilon(x - at, t)$  which satisfies  $u^\epsilon(x, t) \rightarrow u_0(x - at)$  for  $\epsilon \rightarrow 0$ . Assume for simplicity that  $u_0$  is continuous and bounded.

3. Show that for  $u_l > u_r$  the Riemann problem for Burger's equation,

$$u_t + uu_x = 0, \quad u(x, 0) = u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

is solved weakly by the shock  $u(x, t) = u_0(x - st)$  where  $s = (u_l + u_r)/2$  is the shock velocity.

4. Show that

$$u_t + uu_x = \epsilon u_{xx}, \quad u(x, 0) = u_0(x),$$

is solved by  $u^\epsilon(x, t) = w^\epsilon(x - st)$ , where  $s = (u_l + u_r)/2$  and

$$w^\epsilon(x) = u_r + \frac{1}{2}(u_l - u_r)[1 - \tanh((u_l - u_r)x/(4\epsilon))] \begin{cases} \rightarrow u_l, & x \rightarrow -\infty \\ \rightarrow u_r, & x \rightarrow +\infty \end{cases}$$

and show that  $u^\epsilon(x, t) \rightarrow u_0(x - st)$  for  $\epsilon \rightarrow 0$ . Assume for simplicity that  $u_0$  is continuous and bounded.