

# Mathematical Modelling in the Natural Sciences

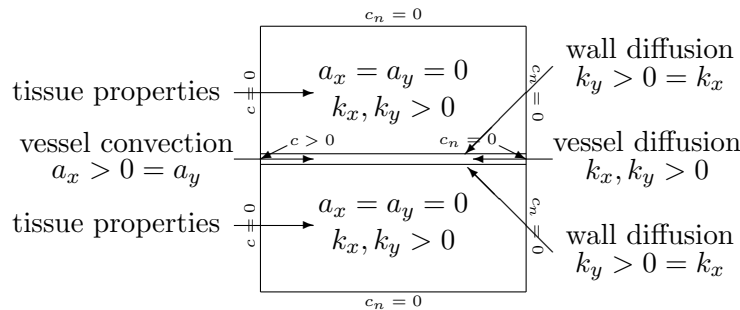
## SS18, Exercises, Sheet 7

Solutions to be presented on 1. June 2018

- Adapt the convection diffusion pseudo-code on pages 145 – 146 in

[http://imsc.uni-graz.at/keeling/numpde\\_ss16/numpde.pdf](http://imsc.uni-graz.at/keeling/numpde_ss16/numpde.pdf)

to write a Matlab code simulating the convection and diffusion of contrast agent in a domain containing a single vessel as follows.



Here, blood flows through the vessel in the  $x$ -direction from left to right with convection coefficients  $a_x > 0 = a_y$  and diffusion coefficients  $k_x, k_y > 0$ . In the tissue outside the vessel, there is no convection so  $a_x = 0 = a_y$ , but there is diffusion with diffusion coefficients  $k_x, k_y > 0$ . There is also diffusion over the boundary of the vessel in the  $y$ -direction with diffusion coefficients  $k_y > 0 = k_x$ . In general, the diffusion coefficients in the vessel, at the vessel wall and in the tissue outside the vessel are different. The boundary conditions for the concentration  $c$  are  $c_n = 0$  on all sides except the left side. On the left side  $c = 0$  holds everywhere except at the vessel inflow where a constant inflow concentration  $c > 0$  holds.

- Let  $c^*$  be the solution from the previous code using known values  $a_x^*, k_x^*, k_y^*$  in the vessel,  $k_y^\dagger$  at the vessel wall and  $k_x^\ddagger, k_y^\ddagger$  in the tissue outside the vessel. Use a perturbation of  $c^*$  as data and write a Matlab code to estimate the known convection and diffusion parameters.
- Verify that the convolution kernel  $K(t)$  for each of the two problems on page 161 of the lecture notes is given in section 2 of <http://imsc.uni-graz.at/invcon/medimage/kernel1.pdf>.
- As discussed on page 162 of the lecture notes, construct a simple example  $(N_\epsilon, C_{\text{AIF}}, E_\epsilon)$  satisfying  $N_\epsilon = C_{\text{AIF}} * E_\epsilon$  and  $N_\epsilon = \mathcal{O}(\epsilon)$  while  $E_\epsilon = \mathcal{O}(\epsilon^{-n})$  for an arbitrary  $n \in \mathbb{N}$ .
- For  $t \in [0, T]$ ,  $T = 4$ , let

$$\begin{aligned} C_{\text{AIF}}(t) &= 20te^{-3t} \\ C_{\text{T}}(t) &= 5e^{-3t}[-5 + 4e^t + e^{2t} - 6t] \\ K(t) &= e^{-t} + e^{-2t} \end{aligned}$$

For  $N \in \mathbb{N}$ , let  $t_i = i\Delta t$ ,  $i = 1, \dots, N$ ,  $\Delta t = T/N$ , let  $\mathbf{C}_{\text{AIF}} = \{C_{\text{AIF}}(t_i)\}_{i=1}^N$ ,  $\mathbf{C}_{\text{T}} = \{C_{\text{T}}(t_i)\}_{i=1}^N$  and  $\mathbf{K} = \{K(t_i)\}_{i=1}^N$ . Then let  $\tilde{\mathbf{C}}_{\text{AIF}}$  and  $\tilde{\mathbf{C}}_{\text{T}}$  be perturbations of  $\mathbf{C}_{\text{AIF}}$  and  $\mathbf{C}_{\text{T}}$ , respectively. For the estimation of  $\mathbf{K}$ , compare the method of truncated singular value decomposition of pages 163 – 165 in the lecture notes with the method of constrained exponentials of pages 167 – 168 in the lecture notes. The Matlab functions `svd` and `lsqlin` are useful for the respective tasks.