

Mathematical Modelling in the Natural Sciences

SS18, Exercises, Sheet 2

Solutions to be presented on 23. March 2018

1. For given data,

$$Y = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

- (a) carry out PCA to determine Y_c , $K = \frac{1}{4}Y_c Y_c^\top$, V and $\Lambda = \text{diag}\{\lambda_i\}$ with $KV = V\Lambda$ and $Y_s = \Lambda^{-\frac{1}{2}}V^\top Y_c$. Plot Y and Y_s .
- (b) Show that the sphered data satisfy $\frac{1}{4}Y_s Y_s^\top = I$. Show that when the sphered data are projected onto an arbitrary axis $\hat{\mathbf{w}} \in \mathbb{R}^2$ with $\|\hat{\mathbf{w}}\|_{\ell_2} = 1$, they have the variance 1.
- (c) For $\theta \in [0, 2\pi]$ define $\mathbf{u} = (\cos(\theta), \sin(\theta))$, $\mathbf{u}^\perp = (-\sin(\theta), \cos(\theta))$ and $U(\theta) = (\mathbf{u}(\theta); \mathbf{u}^\perp(\theta))$. Plot $J(\theta) = \mathcal{K}^2(\mathbf{u}(\theta)Y_s)$ where \mathcal{K} is the kurtosis. Using this plot, determine the value θ^* which maximizes J . Set $X_c = U(\theta^*)Y_s$. Assume that the columns of Y have equal probability, and show that the coordinates (x, y) of columns of the resulting X_c are statistically independent, i.e.,

$$P(x = \alpha \ \& \ y = \beta) = P(x = \alpha) \cdot P(y = \beta).$$

Bonus: Apply PCA to compress the data,

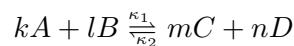
<http://imsc.uni-graz.at/keeling/manuskripten/dcemri.mpg>

down to 10 principal components and thereby denoise the image sequence. Hint: The article

<http://imsc.uni-graz.at/keeling/manuskripten/pcaica.pdf>

may be helpful. Solved by Ms. Stickler.

2. With dimensional analysis derive the third Kepler Law: *The squares of the orbital periods of two planets are proportional to the cubes of their semi-major axes.* (Hint: See page 35 in the script <http://imsc.uni-graz.at/thaller/lehre/mpt/skriptum.pdf>, and don't forget to verify the conditions of the Buckingham Pi Theorem.) Solved by Mr. Berg.
3. The chemical reaction



is given with the following parameters:

$$k = 2, \quad l = 1, \quad m = 2, \quad n = 1$$

$$[A](0) = 2, \quad [B](0) = 2, \quad [C](0) = 2, \quad [D](0) = 1.$$

Determine the initial value problem

$$x'(t) = f(x(t); \kappa_1, \kappa_2), \quad x(0) = x_0$$

where

$$\begin{aligned} [A](t) &= [A](0) - kx(t), & [B](t) &= [B](0) - lx(t) \\ [C](t) &= [C](0) + mx(t), & [D](t) &= [D](0) + nx(t). \end{aligned}$$

Rewrite the equation $f(x^*; \kappa_1, \kappa_2) = 0$ in the form,

$$\kappa_2/\kappa_1 = r(x^*)$$

giving a relationship between equilibria and the quotient κ_2/κ_1 . (Note that the condition $\kappa_2/\kappa_1 > 0$ implies, through the form of r , that an equilibrium must satisfy $x^* > -1$.) Show that any equilibrium x^* with $r'(x^*) < 0$ is (locally asymptotically) stable, while any equilibrium x^* with $r'(x^*) > 0$ is unstable. (Hint: It holds that $f'(x^*; \kappa_1, \kappa_2)/\kappa_1 = 4r'(x^*)(1 + x^*)^3$.) Derive a corresponding potential landscape $p(x, \kappa_2/\kappa_1)$, where $f(x; \kappa_1, \kappa_2)/\kappa_1 = -p_x(x, \kappa_2/\kappa_1)$. For various values of κ_2/κ_1 , plot $f(x; \kappa_1, \kappa_2)$ and $p(x, \kappa_2/\kappa_1)$ in the interval $0 \leq x \leq 3$ to demonstrate the associated Hysteresis graphically. Solved by Mr. Steuber.