Mathematical Modelling in the Natural Sciences
SS17, Exercises, Sheet 4

Solutions to be presented on 30. March 2017

1. Let $p_n(t) = P\{X(t) = n\}$ be the probability that n customers are waiting to be served at time t, and there are two cashiers. The system of ODEs for these probabilities is:

$$\begin{aligned} p_0'(t) &= -b_0 p_0(t) + d_1 p_1(t) \\ p_n'(t) &= b_{n-1} p_{n-1}(t) - (b_n + d_n) p_n(t) + d_{n+1} p_{n+1}(t), \quad 1 \le n \le N-1 \\ p_N'(t) &= b_{N-1} p_{N-1}(t) - d_N p_N(t) \end{aligned}$$

where the coefficients $\{b_n\}$ and $\{d_n\}$ are determined as follows.

- The average time between customer arrivals is $c = 1/b_n$ and is independent of the number of cashiers.
- If there is only one customer, then $s = 1/d_1$ is the average service time when only one cashier is in operation.
- When there are at least two customers, then $s/2 = 1/d_n$, $2 \le n \le N$, is the average service time when two cashiers are in operation.

Therefore it holds

$$b_n = 1/c, \quad d_n = \begin{cases} 2/s, & 2 \le n \le N \\ 1/s, & n = 1 \end{cases}$$

Let $\{p_n^\star\}$ be the stationary state for $X(t)$. Show with $\rho = s/c$,

$$E[X^\star] = p_0 \rho \frac{N(\rho/2)^{N+1} - (N+1)(\rho/2)^N + 1}{(1-\rho/2)^2}, \quad p_0 = \frac{1-\rho/2}{1+\rho/2-\rho(\rho/2)^N}$$

and

$$E[X^\star] \stackrel{\rho\to 2}{\longrightarrow} \frac{N(N+1)}{1+2N}, \quad E[X^\star] \stackrel{N\to\infty}{\longrightarrow} \frac{4\rho}{4-\rho^2} \equiv L_2(\rho) \quad \text{für } \rho \in (0,2)$$

2. Define a random variable X where $X(n) = i$ if an elevator is at the i-th floor after n time intervals Δt. Suppose $i \in \{0(\mathrm{G}), 1, \dots, N\}$. For $p_i(n) = P(X(n) = i)$ and $\boldsymbol{p}(n) = \{p_i(n)\}_{i=0}^N$, let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be a stochastic matrix containing transition probabilities satisfying $\boldsymbol{p}(n) = P^\top \boldsymbol{p}(n-1)$. Assume that Δt is small enough that jumps of two floors with a time interval Δt is impossible, and hence P is tridiagonal but otherwise arbitrary; in particular, P need not be symmetric. For some $N \in \mathbb{N}$ and for sample entries in P, determine the stationary (equilibrium) distribution of states $\boldsymbol{p}^\star$ for the elevator and demonstrate by simulation that $\boldsymbol{p}_0^\top P^n \to \boldsymbol{p}^{\star\top}$, $n \to \infty$, for any starting distribution $\boldsymbol{p}_0$. Show also that $P^n > 0$ for some n, and that all rows of P^n converge to $\boldsymbol{p}^{\star\top}$. Under which circumstances are all entries of $\boldsymbol{p}^\star$ the same?

3. For the parameters $\beta = 100$, $\mu = 0.001$, $\gamma = 0.4$ and $\lambda = 5 \cdot 10^{-6}$ implement the SIR model,

$$S' = \beta - (\mu + \lambda I)S, \quad I' = (\lambda S - \mu - \gamma)I, \quad R' = -\mu R + \gamma I$$

and plot the results in time and in phase space. Prove that the equilibrium obtained is locally asymptotically stable.