Mathematical Modelling in the Natural Sciences
SS16, Exercises, Sheet 10

Solutions to be presented on 3. June 2016

1. Let $\bar{\Omega} = [0,1]$ and $\tilde{u}(x) = \text{sign}(x - \frac{1}{2})$, and consider the minimization of the functional,

$$J(u) = \int_0^1 |u - \tilde{u}|^2 dx + \mu \text{TV}_\Omega(u)$$

where, for u sufficiently smooth, $\text{TV}_\Omega(u) = \|u'\|_{L^1(\Omega)}$,[1] but otherwise

$$\text{TV}_\Omega(u) = \sup_{\{x_i\}_{i=0}^{N+1} \subset \Omega, x_i > x_{i-1}} \sum_{i=1}^{N+1} |u(x_i) - u(x_{i-1})|.$$

(a) Let $\text{BV}(\Omega)$ denote the set of functions with bounded variation in Ω equipped with the norm $\|u\|_{\text{BV}} = \|u\|_{L^1(\Omega)} + \text{TV}_\Omega(u)$. With Hölder's Inequality,[2] show that if $J(u) < \infty$ holds, then u satisfies

$$\|u\|_{L^1(\Omega)} = \|1 \cdot u\|_{L^1(\Omega)} \leq \quad \ldots \text{fill in the details} \ldots \quad \leq \sqrt{J(u)} + \|\tilde{u}\|_{L^2(\Omega)} < \infty$$

as well as

$$\text{TV}_\Omega(u) \leq J(u)/\mu < \infty.$$

Hence $\|u\|_{\text{BV}} < \infty$ and $u \in \text{BV}(\Omega)$. Thus, minimization of J is performed over the set of functions in $\text{BV}(\Omega)$.

(b) Suppose $u^* \in \text{BV}(\Omega)$ is a minimizer for J. Set

$$\bar{u}(x) = \begin{cases} \max\{-1, \operatorname*{essinf}_{x \in [0,\frac{1}{2}]} u^*(x)\}, & x \in [0, \frac{1}{2}] \\ \\ \min\{+1, \operatorname*{esssup}_{x \in (\frac{1}{2},1]} u^*(x)\}, & x \in (\frac{1}{2}, 1] \end{cases}$$

and show that $J(\bar{u}) \leq J(u^*)$.[3] Thus, for the particular case that $\tilde{u} = \text{sign}(x - \frac{1}{2})$, minimization of J is performed over the two-dimensional set of functions

$$u(x; a, b) = a \cdot \text{sign}(x - \tfrac{1}{2}) + b, \quad -1 \leq b - a, \quad a + b \leq 1.$$

(c) Show that for μ sufficiently small, the minimizer $u_\mu^\star$ satisfies $u_\mu^\star = (1 - \mu)\tilde{u}$.

(d) Show that for μ sufficiently large, the minimizer $u_\mu^\star$ satisfies $u_\mu^\star = 0$.

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[1] To be precise, note that $L^p(\Omega)$ consists of the Lebesgue measurable functions u for which $|u|^p$ is integrable on Ω.

[2] Hölder's Inequality: $\forall f \in L^p(\Omega)$, $\forall g \in L^q(\Omega)$, with $1/p + 1/q = 1$, it holds that $\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$.

[3] To be precise, note that $\text{esssup}_{x \in D}\, u(x)$ is the supremum or least upper bound of values $\{u(x) : x \in \tilde{D}, |D \backslash \tilde{D}| = 0\}$ where $|D|$ denotes the Lebesgue measure of the set D. Similarly, $\text{essinf}_{x \in D}\, u(x) = \inf\{u(x) : x \in \tilde{D}, |D \backslash \tilde{D}| = 0\}$ where inf denotes the greatest lower bound.

(e) Develop a method to minimize the discretization with respect to $\vec{u} = \{u_i\}_{i=0}^{N+1}$,

$$J_h(\vec{u}) = h \sum_{i=0}^{N+1} |u_i - \tilde{u}_i|^2 + \mu \sum_{i=1}^{N+1} |u_i - u_{i-1}|$$

where $\{\tilde{u}_i\}_{i=0}^{N+1}$ are noisy data and $u_i \approx u(x_i)$, $x_i = ih$, $h = 1/(N+1)$. Implement your method with Matlab. Hint: See pages 183 – 193 in

`http://imsc.uni-graz.at/keeling/num1_ws14/numerik.pdf`

2. Implement the code from the lecture notes to simulate large deformations of a bungee cord. Explain how to reformulate the non-linear wave equation in first order form so that the norm of the state corresponds to conserved energy. Does the Crank Nicholson scheme conserve this energy with each time step?

3. Derive the necessary stationarity conditions on page 202 of the lecture notes for the wave equation modelling large deformations in a membrane. A code to solve this problem numerically is given at

`http://imsc.uni-graz.at/keeling/numpde_ss16/sheet.m`

4. Minimization of Laplacian energy: Let $u^* = \{u_i^*\}_{i=0}^{N}$ be a discrete signal on the interval $[0, 1]$, where the data points $\{x_i\}_{i=0}^{N}$ are uniformly distributed on the interval, i.e., u_i^* corresponds to the value $u^*(x_i)$ for $x_i = i/N$. However, noise is added to u on the interior $(0, 1)$ and not on the boundary $\{0, 1\}$, which results in the data $\tilde{u} = \{u_0^*, \{\tilde{u}_i\}_{i=1}^{N-1}, u_N^*\}$. In practice, one starts with $\tilde{u}$ as an estimate of u^* and aims at reducing the noise in the difference $u^* - \tilde{u}$, thereby retrieving an estimate of u^*. As motivation for the subsequent method let us observe the problem

$$\hat{u} \in \underset{u}{\operatorname{argmin}} \frac{1}{2} \|\Delta_h u\|_2^2 = J(u), \quad \text{subject to } u_0 = \tilde{u}_0 \text{ and } u_N = \tilde{u}_N, \tag{1}$$

where $(\Delta_h u)_i = u_{i-1} - 2u_i + u_{i+1}$ for $i = 1, \ldots, N-1$. The discrete Laplacian operator $\Delta_h u$ has some relation to the curvature of objects and thus reducing the norm of $\Delta_h u$ is expected to reduce oscillations.

(a) Derive a formula for the solution to (1) and show a solution exists and it is unique. (Hint: Reformulate the functional as the norm of a linear operation A applied to $u_1, \cdots, u_{N-1}+$ a constant term, i.e., $J(u) = \|A(u_1 ... u_{N-1}) + b\|_2^2$.)

(b) How can you characterize the solution with respect to $\tilde{u}_0$ and $\tilde{u}_N$ geometrically? Does the solution to this problem necessarily correspond to the original data u^* we are looking for? (Hint: Use induction or a uniqueness argument to prove the characterization.)

After taking these theoretical arguments into account, one might think there is actually no point whatsoever in looking at this problem. In order to understand why it is still reasonable to look at this problem, we solve the problem iteratively. In order to solve this problem numerically we look at the following very simple procedure for a given $\tilde{u}$,

$$\begin{aligned} &u_0 = \tilde{u}, \quad n = 1, \ldots, M, \quad r \in (0, 1) \\ &u_n(i) = (1 - r) * u_n(i-1) + \frac{r}{2}\big((u_{n-1}(i) + u_{n-1}(i+1)\big). \end{aligned} \tag{2}$$

Since one iteration of this scheme usually is not enough to change the data sufficiently, we repeat the method for a fixed number of iterations denoted by M.

- Implement the scheme (2).
- Look at the problem $u^*(x) = \text{sign}(x - 1/2)$ for (the original signal) and add noise to $u_1, \ldots, u_{N-1}$ (the noise `0.1*randn(1,N-1)` is reasonable) in order to obtain $\tilde{u}$. Try to retrieve u^* for $N = 100$ solving numerically with $r = 0.2$ and $M = 30$ and using the scheme (2). Plot $\tilde{u}$,u_n and u^* after each iteration in order to visualize the developments. Surprised about the results?
- Try to play around with the parameter r and M for the example above in order to understand the behavior.
- Let the code run for very large M. Does it converge towards the derived minimum? Also investigate what happens for $r > 1$, in particular for $r > 2$?