

# Mathematical Modelling in the Natural Sciences

## SS16, Exercises, Sheet 4

*Solutions to be presented on 14. April 2016*

- Let  $p_n(t) = P\{X(t) = n\}$  be the probability that  $n$  customers are waiting to be served at time  $t$ , and there are two cashiers. The system of ODEs for these probabilities is:

$$\begin{aligned} p_0'(t) &= -b_0 p_0(t) + d_1 p_1(t) \\ p_n'(t) &= b_{n-1} p_{n-1}(t) - (b_n + d_n) p_n(t) + d_{n+1} p_{n+1}(t), \quad 1 \leq n \leq N-1 \\ p_N'(t) &= b_{N-1} p_{N-1}(t) - d_N p_N(t) \end{aligned}$$

where the coefficients  $\{b_n\}$  and  $\{d_n\}$  are determined as follows.

- The average time between customer arrivals is  $c = 1/b_n$  and is independent of the number of cashiers.
- If there is only one customer, then  $s = 1/d_1$  is the average service time when only one cashier is in operation.
- When there are at least two customers, then  $s/2 = 1/d_n$ ,  $2 \leq n \leq N$ , is the average service time when two cashiers are in operation.

Therefore it holds

$$b_n = 1/c, \quad d_n = \begin{cases} 2/s, & 2 \leq n \leq N \\ 1/s, & n = 1 \end{cases}$$

Let  $\{p_n^*\}$  be the stationary state for  $X(t)$ . Show with  $\rho = s/c$ ,

$$E[X^*] = p_0 \rho \frac{N(\rho/2)^{N+1} - (N+1)(\rho/2)^N + 1}{(1 - \rho/2)^2}, \quad p_0 = \frac{1 - \rho/2}{1 + \rho/2 - \rho(\rho/2)^N}$$

and

$$E[X^*] \xrightarrow{\rho \rightarrow 2} \frac{N(N+1)}{1+2N}, \quad E[X^*] \xrightarrow{N \rightarrow \infty} \frac{4\rho}{4-\rho^2} \equiv L_2(\rho) \quad \text{für } \rho \in (0, 2)$$

(Solved by Filipe Martinho)

- Define a random variable  $X$  where  $X(n) = i$  if an elevator is at the  $i$ -th floor after  $n$  time intervals  $\Delta t$ . Suppose  $i \in \{0(G), 1, \dots, N\}$ . For  $p_i(n) = P(X(n) = i)$  and  $\mathbf{p}(n) = \{p_i(n)\}_{i=0}^N$ , let  $P \in \mathbb{R}^{(N+1) \times (N+1)}$  be a stochastic matrix containing transition probabilities satisfying  $\mathbf{p}(n) = P^\top \mathbf{p}(n-1)$ . Assume that  $\Delta t$  is small enough that jumps of two floors with a time interval  $\Delta t$  is impossible, and hence  $P$  is tridiagonal but otherwise arbitrary; in particular,  $P$  need not be symmetric. For some  $N \in \mathbb{N}$  and for sample entries in  $P$ , determine the stationary (equilibrium) distribution of states  $\mathbf{p}^*$  for the elevator and demonstrate by simulation that  $\mathbf{p}_0^\top P^n \rightarrow \mathbf{p}^{*\top}$ ,  $n \rightarrow \infty$ , for any starting distribution  $\mathbf{p}_0$ . Show also that  $P^n > 0$  for some  $n$ , and that all rows of  $P^n$  converge to  $\mathbf{p}^{*\top}$ . Under which circumstances are all entries of  $\mathbf{p}^*$  the same? (Solved by Andreas Holm)
- For the parameters  $\beta = 100$ ,  $\mu = 0.001$ ,  $\gamma = 0.4$  and  $\lambda = 5 \cdot 10^{-6}$  implement the *SIR* model,

$$S' = \beta - (\mu + \lambda I)S, \quad I' = (\lambda S - \mu - \gamma)I, \quad R' = -\mu R + \gamma I$$

and plot the results in time and in phase space. Prove that the equilibrium obtained is locally asymptotically stable. (Solved by Florian Thaler)