

Mathematical Modelling in the Natural Sciences

SS16, Exercises, Sheet 3

Solutions to be presented on 17. March 2016

1. Show that the equilibrium $(x^*, y^*) = (a_2/b_2, a_1/b_1)$ for the Gause equations,

$$x'(t) = (a_1 - b_1y)x, \quad y'(t) = (a_2 - b_2x)y, \quad a_1, a_2, b_1, b_2 > 0$$

is unstable and, in particular, a saddle point. (Solved by Ornella Moro)

2. Let the damped mass spring model,

$$mu''(t) = f - ku(t) - cu'(t)$$

for $m = 1$, $k = 1$, $f = 1$ and $c = 3$ be written in first order form

$$\mathbf{u}'(t) = A\mathbf{u}(t) + \mathbf{b}, \quad \mathbf{u}(t) = (u(t); u'(t)), \quad A = [0, 1; -k, -c]/m, \quad \mathbf{b} = (0; f/m).$$

First show that the equilibrium

$$\mathbf{u}^* = (u^*; 0) = -A^{-1}\mathbf{b}$$

is locally asymptotically stable. Then note that the system can also be written as

$$\mathbf{w}'(t) = A\mathbf{w}(t), \quad \mathbf{w}(t) = \mathbf{u}(t) - \mathbf{u}^*.$$

To show that $\mathbf{w} = \mathbf{0}$ is globally asymptotically stable, derive a function $P(\mathbf{w})$ satisfying

$$S^{-\top}S^{-1}A\mathbf{w} = -\nabla P(\mathbf{w}), \quad AS = S\Lambda, \quad \Lambda = \text{diag}\{\lambda_i\}_{i=1}^2$$

and show that P decreases to its global minimum at $\mathbf{w} = \mathbf{0}$ along every solution $\mathbf{w}(t)$. (Solved by Andreas Holm)

3. Show that the function

$$F(x, y) = -a_2 \ln(x) + b_2x - a_1 \ln(y) + b_1y$$

is a Lyapunov function for the predator-prey model,

$$x' = (a_1 - b_1y)x, \quad y' = (b_2x - a_2)y, \quad a_1, a_2, b_1, b_2 > 0.$$

(Solved by Florian Thaler)