Study Questions for Mathematical Image Processing
Wintersemester 2009/10

1. For $\Omega = (0,1)$ and $0 < \alpha < \frac{1}{2}$, define the following functional on $W^{1,1}(\Omega)$,

$$F(u) = \int_\Omega [|u(x) - \chi(x)| + \alpha |u'(x)|]\, dx, \quad \text{where} \quad \chi(x) = \begin{cases} 0, & x < \frac{1}{2} \\ 1, & x \geq \frac{1}{2} \end{cases}$$

   Show that

$$\inf_{u \in W^{1,1}(\Omega)} F(u) = \alpha$$

   and construct a minimizing sequence $u_\epsilon \in W^{1,1}(\Omega)$, i.e., $F(u_\epsilon) \to \alpha$, $\epsilon \to 0$. Show that the sequence has no accumulation in $W^{1,1}(\Omega)$. On the other hand, find a minimizer in $W^{1,1}(\Omega)$ for the case that $\alpha \geq \frac{1}{2}$.

2. Suppose that $\tilde{u}$ is an image measured on a domain $\Omega \subset \boldsymbol{R}^N$ where $\tilde{u} = u^\star + \nu$ for an exact underlying image $u^\star$ perturbed with normally distributed noise ν. The image $u^\star$ is estimated from the measurement $\tilde{u}$ by minimizing the functional:

$$J(u) = \int_\Omega [u - \tilde{u}]^2 dx + \int_\Omega \phi(|\nabla u|) dx$$

   For instance, $\phi(s) = \alpha s^p$, $\alpha > 0$, $p \geq 1$, is a possibility. Show that when $u \in C^2(\Omega)$ is a minimizer, then u necessarily satisfies:

$$\begin{cases} -\nabla \cdot \left[\dfrac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \right] + u &= \tilde{u}, \quad \Omega \\ \partial u / \partial n &= 0, \quad \partial \Omega \end{cases}$$

3. Consider the case in problem 2 that $\Omega = (0,1) \in \boldsymbol{R}^1$ is discretized by the grid with cells having centers $x_i = h(i - 1/2)$, $i = 1, \ldots, n$, $h = 1/n$. Assume that $\phi(s) = \alpha s^2$ with $\alpha > 0$. Develop a symmetric discretization of the optimality condition and write a MATLAB code to solve it for a given $\tilde{u}$ and α. As a test case, let $u^\star$ be a step function and set $\tilde{u} = u^\star + \nu$ according to a certain noise level ν. Then determine the parameter α for which the numerical reconstruction comes closest to $u^\star$.

4. Newton's method for minimizing J in problem 2 takes the form:

$$\begin{cases} \dfrac{\delta^2 J}{\delta u^2}(u_k; \bar{u}, \delta u) &= -\dfrac{\delta J}{\delta u}(u_k; \bar{u}), \quad \forall \bar{u} \in C^\infty(\boldsymbol{R}^N) \\ u_{k+1} &= u_k + \sigma \delta u \end{cases} \qquad k = 1, 2, \ldots$$

   for a given step size $\sigma > 0$. By neglecting ϕ'', a modified Newton's method is given by:

$$\begin{cases} -\nabla \cdot \left[\dfrac{\phi'(|\nabla u_k|)}{|\nabla u_k|} \nabla \delta u \right] + \delta u &= \nabla \cdot \left[\dfrac{\phi'(|\nabla u_k|)}{|\nabla u_k|} \nabla u_k \right] - u_k + \tilde{u}, \quad \Omega \\ \partial \delta u / \partial n &= 0, \quad \partial \Omega \\ u_{k+1} &= u_k + \sigma \delta u, \quad k = 1, 2, \ldots \end{cases}$$

   For $\Omega = (0,1) \in \boldsymbol{R}^1$ develop a symmetric discretization of this problem. (Hint: Let ∇_h be a (forward or backward difference) discretization of the gradient which maps cell center values to cell interface values. Then $-\nabla_h^{\mathrm{T}}$ is a discretization of the divergence operator. Also $\phi'(|\nabla_h u|)/|\nabla_h u|$ is a vector of coefficient values at cell interfaces. Finally, $\nabla_h^{\mathrm{T}} \mathrm{diag}(\phi'(|\nabla_h u|)/|\nabla_h u|) \nabla_h \delta u$ is a discretization of the differential opeator above.) Write a MATLAB code to implement this method for $\phi(s) = \alpha s^p$, $\alpha > 0$, $p \geq 1$. (Take care in your code to set a term to zero when its denominator may be zero.) As a test case, let $u^\star$ be a step function and set $\tilde{u} = u^\star + \nu$ according to a certain noise level ν. Then determine the parameters α and p for which the numerical reconstruction comes closest to $u^\star$.

5. Define the Bolza functional,

$$F(u) = \int_\Omega [u(x)^2 + (u'(x)^2 - 1)^2]dx, \quad u \in W_0^{1,4}(\Omega), \quad \Omega = (0,1)$$

Show that on $W_0^{1,4}(\Omega)$, (a) F is not convex, (b) F is not lower semicontinuous with respect to the weak topology and F does not admit a minimizer.

6. Determine the polar and bipolar functions for (a) $f(x) = |x|$ and (b) $f(x) = (x^2 - 1)^2$.

7. For $\{a_n\} \subset L^\infty(\Omega)$ satisfying

$$0 < c_1 \le a_n(x) \le c_2, \quad \text{a.e.} x \in \Omega$$

as well as

$$a_n \overset{\star}{\underset{L^\infty(\Omega)}{\rightharpoonup}} a \in L^\infty(\Omega), \quad 1/a_n \overset{\star}{\underset{L^\infty(\Omega)}{\rightharpoonup}} b \in L^\infty(\Omega)$$

define the functionals,

$$F_n(u) = \int_\Omega a_n(x) u(x)^2 dx, \quad u \in L^2(\Omega), \quad n = 1, 2, \dots$$

Also define the functionals

$$F(u) = \int_\Omega u(x)^2/b(x)dx, \quad G(u) = \int_\Omega a(x)u(x)^2 dx, \quad u \in L^2(\Omega)$$

Although $F = \Gamma - \lim_{n\to\infty} F_n$ for $\tau =$ weak topology of $L^2(\Omega)$, show that $G = \Gamma - \lim_{n\to\infty} F_n$ for $\tau =$ strong topology of $L^2(\Omega)$.

8. Show that the Hausdorf dimension of the Cantor set is $\log 2 / \log 3$.

9. For $\Omega \subset\subset \boldsymbol{R}^n$ and $\phi : \boldsymbol{R} \to \boldsymbol{R}$ convex and satisfying $\phi(s) \le k(1+|s|)$, $\forall s \in \boldsymbol{R}$, show that $F(u) = \int_\Omega \phi(|Du|)$ is lower semicontinuous with respect to the strong topology on $L^1(\Omega)$ (and hence the weak-$\star$ topology on $\mathrm{BV}(\Omega)$). Hint: For $g : \boldsymbol{R}^n \to \boldsymbol{R}$ convex and satisfying $g(\boldsymbol{\xi}) \le k(1 + |\boldsymbol{\xi}|)$, $\forall \boldsymbol{\xi} \in \boldsymbol{R}^n$, $G(u) = \int_\Omega g(Du)$ can be written as [Ref 130]:

$$G(u) = \sup_{\boldsymbol{\xi} \in \mathcal{D}_{g^*}} \int_\Omega \boldsymbol{\xi} \cdot Du - \int_\Omega g^*(\boldsymbol{\xi}) dx$$

where $\mathcal{D}_{g^*} = \{\boldsymbol{\xi} \in C_0^\infty(\Omega, \boldsymbol{R}^n) : g^* \circ \boldsymbol{\xi} \in L^1(\Omega)\}$.

10. For $J(u) = \int_\Omega [Ru - u_0]^2 dx$, $J : L^2(\Omega) \to \boldsymbol{R}$, $R \in \mathcal{L}(L^2(\Omega))$,

$$[Ru](x) = \int_\Omega \rho(x-y)u(y)dy, \quad \int_{\boldsymbol{R}^n} \rho(x)dx = 1, \quad \rho \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{R})$$

derive the necessary optimality condition for a minimizer of J:

$$R^* Ru = R^* u_0$$

11. Consider the case in the last problem that $\Omega = (0,1)$ ($N = 1$) is discretized by the grid with cells having centers $x_i = h(i - 1/2)$, $i = 1, \dots, n$, $h = 1/n$. Also suppose $\rho \in C^0(\boldsymbol{R})$ is given by:

$$\rho(z) = \begin{cases} 1/(2h), & z \in [-h, h] \\ 0, & \text{else} \end{cases}$$

and show that with an appropriate integration rule, Ru can be approximated by:

$$[Ru](x_i) \approx \frac{u_{i+1} + 2u_i + u_{i-1}}{4}, \quad 1 < i < N, \quad [Ru](x_1) \approx \frac{3u_1 + u_2}{4}, \quad [Ru](x_n) \approx \frac{u_{n-1} + 3u_n}{4}$$

Denote this discretization by $[Ru](x_i) \approx (R_h U)_i$ where $U = \{u_i\}$. Write a MATLAB code to solve the discretized normal equations $R_h^{\mathrm{T}} R_h U = R_h^{\mathrm{T}} \tilde{U}$. Then consider the following spectral approach to estimating $u^\star$ from $\tilde{u}$. Let $R_h = PSQ^{\mathrm{T}}$ be the singular value decomposition of R_h. Define a regularized estimator by

$$U_\alpha = QS_\alpha^{-1} P^{\mathrm{T}} \tilde{U}$$

where:

$$S_\alpha^{-1} = \mathrm{diag}\left\{ \frac{(\sigma_i > \alpha \cdot \bar{\sigma})}{\sigma_i} \right\}, \quad \bar{\sigma} = \max\{\sigma_i\}$$

Write a MATLAB code to implement this spectral estimation method and determine the value of α for which the reconstruction U comes closest to $U^\star$.

12. Show for the 2D case that

$$\nabla \cdot \left[\frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \right] = \frac{\phi'(|\nabla u|)}{|\nabla u|} u_{\tau\tau} + \frac{\phi''(|\nabla u|)}{|\nabla u|} u_{\nu\nu}$$

where $u_{\tau\tau} = \tau^{\mathrm{T}} D^2 u \tau$, $\tau = (u_y, -u_x)/|\nabla u|$, and $u_{\nu\nu} = \nu^{\mathrm{T}} D^2 u \nu$, $\tau = (u_x, u_y)/|\nabla u|$.

13. Verify the claim on p. 61 of the textbook with respect to the Poincáre-Wirtinger inequality: *Observe that the same inequality holds for functions of bounded variation, where $|\nabla u|_{L^p(\Omega)}$ is replaced by the total variation $|Du|(\Omega)$.* (Hint: First note that the claim is not true unless the $L^p(\Omega)$ on the left is restricted to $p \le n/(n-1)$ for $\Omega \subset \boldsymbol{R}^n$. Then modify Theorem 2 on p. 172 of Ref. [149] to obtain: *For $u \in BV(\Omega) \cap L^2(\Omega)$ there exists $\{u_k\}_{k=1}^\infty \subset BV(\Omega) \cap C^\infty(\Omega)$ such that $u_k \to u$ in $L^2(\Omega)$ and $|Du_k|(\Omega) \to |Du|(\Omega)$ as $k \to \infty$.*)

14. Let the conditions on the function ϕ and the operator R in Theorem 3.2.2 be satisfied. For $V = \{u \in L^2(\Omega) : \nabla u \in L^1(\Omega, \boldsymbol{R}^n)\}$ define:

$$E(u) = \begin{cases} \frac{1}{2} \int_\Omega [Ru - u_0]^2 dx + \lambda \int_\Omega \phi(|\nabla u|) dx, & u \in V \\ +\infty, & \text{else} \end{cases}$$

By replacing ∇u with Du and V with $\mathrm{BV}(\Omega)$ define:

$$\bar{E}(u) = \begin{cases} \frac{1}{2} \int_\Omega [Ru - u_0]^2 dx + \lambda \int_\Omega \phi(|Du|) dx, & u \in \mathrm{BV}(\Omega) \\ +\infty, & \text{else} \end{cases}$$

According to Theorem 3.2.1, $\bar{E} = R_\tau E$, where $\tau =$ weak $\star$ topology on $\mathrm{BV}(\Omega)$. Let $u^\star$ be the unique minimizer of $\bar{E}$ in $\mathrm{BV}(\Omega)$, guaranteed by Theorem 3.2.2. Show that $\bar{E}(u^\star) \le \inf_{v \in V} E(v)$. (Hint: Consider Theorem 2.1.5.) Does equality hold? (Hint: Consider Theorem 2.1.6 after showing $\bar{E} = R_\sigma E$, where σ is the strong topology in $L^2(\Omega)$.)

15. Define $\tilde{G} = \{v = \nabla \cdot p : |p| \le 1, p \in C_0^1(\Omega, \boldsymbol{R}^N)\}$ and $G = \{v = \nabla \cdot p : |p| \le 1, p \in H_0(\mathrm{div})\}$ where $H_0(\mathrm{div}) = \{p \in L^2(\Omega, \boldsymbol{R}^N) : \nabla \cdot p \in L^2(\Omega), p \cdot n = 0, \partial\Omega\}$. Show that G is the closure of $\tilde{G}$ with respect to the $L^2(\Omega)$ topology.

16. Show that χ_G, the indicator function for G above, is convex, lower semicontinuous and proper on $L^2(\Omega)$.

17. Show that the convex conjugate of χ_G satisfies $\chi_G^*(u) = \mathrm{TV}(u) = |Du|(\Omega)$.

18. Let the conditions on the function ϕ and the operator R in Theorem 3.2.2 be satisfied. Let ϕ_ϵ be the *half quadratic* approximation of the function ϕ as defined on page 80. Define

$$E_\epsilon(u) = \begin{cases} \frac{1}{2} \int_\Omega |Ru - u_0|^2 dx + \lambda \int_\Omega \phi_\epsilon(|\nabla u|), & u \in W^{1,2}(\Omega) \\ +\infty, & \text{else} \end{cases}$$

Show there is a unique $u_\epsilon \in W^{1,2}(\Omega)$ such that $E_\epsilon(u_\epsilon) = \min$.

19. With the understanding that $\phi_0 = \phi$ in problem 18, show that $\forall u \in W^{1,2}(\Omega)$, $E_\epsilon(u) \searrow E_0(u)$ as $\epsilon \to 0$.

20. Let $\bar{E}$ be as defined in problem 14 and let E_0 be as defined in problem 19. Show that $\forall u \in \mathrm{BV}(\Omega)$, $\exists \{u_n\} \subset W^{1,2}(\Omega)$ such that $u_n \stackrel{L^1(\Omega)}{\longrightarrow} u$ and $\bar{E}(u) = \liminf_{n\to\infty} E_0(u_n)$. (Hint: With the standard mollifier

$$\rho(x) = \begin{cases} C\exp(1/(|x|^2-1)), & |x| < 1 \\ 0, & \text{else} \end{cases} \qquad \int_{\boldsymbol{R}^n} \rho(x)dx = 1, \quad \rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$$

combine the techniques of Theorem 2 on p. 172 of Ref. [149] and of Lemma 2.2 on page 685 of Ref. [130], `http://math.uni-graz.at/keeling/convexmeas.pdf`.)