On Approximation in Total Variation Penalization for Image Reconstruction and Inverse Problems*

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Abstract. In this paper, we examine some theoretical issues associated with the use of total variation based image reconstruction. Our investigations are motivated by problems of inverse interferometry, in which laser light phase shifts are used to reconstruct medium density profiles in flow field sensing. The reconstruction problem is posed as a residual minimization with total variation regularization applied to handle the inherent ill-posedness. We consider numerical approximations of these penalized minimal residual problems, and we analyze some approximation strategies and their properties. The standard definition of total variation leads to inconsistent approximations with piecewise constant basis functions, so we consider alternative definitions which preserve the needed compactness and produce convergent approximations.

1 Introduction and Motivation

In this paper, we consider image reconstruction techniques via residual minimization with total variation penalization. Such methods are becoming rather widely used in a variety of applications; see, e.g., [4], [6], [8], [15], [18]. While much is known in terms of algorithms and theory, there are still some issues to be addressed associated with numerical approximations and convergence. We focus here on the use of piecewise constant elements for approximating the image to be reconstructed, and we analyze solutions of approximate problems with respect to numerical grid refinement. Crucial issues which we address are the computation and approximation of the total variation and of L^1 norms involved in the minimum residual approach.

1.1 The Basic Problem

The general problem to be considered involves a theoretical image $f \in X$, an observed image $g \in Z$, and an observation operator $\Phi : X \to Z$. For instance, $X = L^1(F)$, where F is a bounded flow-field domain in $\mathbb{R}^d$, and $Z = L^1(P)$, where P is a bounded measurement domain in $\mathbb{R}^2$. Alternatively, Z may be a finite dimensional space of *pixel values.* A typical problem involves solving for f in $\Phi f = g$ with observed data $g \in Z$. The problem is treated by minimizing some measure of the residual, $\Phi f - g$, subject to smoothing constraints which *de-noise* the image f. In this paper, we consider smoothing via the total variation of the image.

We remark here that the use of the L^1 residual criterion arises for several reasons. One is that L^1 is a natural space for flow-field densities, and the operator Φ lies in $\mathcal{L}(L^1(F), L^1(P))$, as shown below. Another is that the effectiveness of smoothing via the total variation requires BV to be compactly imbedded in X, a condition met by $X = L^1$ but not $X = L^2$ in dimension $d > 2$ [7]. Finally, the L^1 criterion is typically more robust in a statistical sense than the L^2 criterion; see, e.g., [3].

A penalized minimal residual functional of particular interest is given by

$$J_\mu(f) = \|\Phi f - g\|_Z + \mu TV(f),$$

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where the total variation, $TV(f)$, is given by the formula

$$TV(f) = \sup_{\Psi \in \mathcal{C}} \int_F f \nabla \cdot \Psi,$$

with

$$\mathcal{C} = \{\Psi : F \to \mathbb{R}^d, \Psi \in C_0^1, |\Psi|_{\ell_2} \leq 1\}.$$

Here, C_0^1 denotes continously differentiable functions that vanish on the boundary of F, and $|\cdot|_{\ell_p}$ is the usual p norm in $\mathbb{R}^d$. In one dimension, the total variation is also given by

$$TV(f) = \sup \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})|$$

where the supremum is taken over all finite partitions $\{x_k\}$. It is also important to note that if the function f has integrable partial derivatives, then

$$TV(f) = \int_F |\nabla f|_{\ell_2}.$$

In general, however, functions of bounded variation need not be continuous, which is a driving reason behind the use of total variation penalty terms.

The motivation for the total variation penalty, as opposed to a simpler Hilbert space penalty such a Sobolev $W^{1,2}$ norm, is the level of smoothness imposed. For example, characteristic functions of sets having finite perimeter have bounded variation but not a finite $W^{1,2}$ norm. However, in most cases the total variation norm is strong enough to provide the compactness needed in ill-posed problems, as we shall see below.

In order to minimize J_μ, several approximations must be made. First, one must consider a finite dimensional subspace of approximating “image functions.” As we discuss below, we use piecewise constant approximations. Another approximation we make (as in [1]) involves slight perturbations of the L^1 and the TV functionals in order to have a differentiable cost functional. Our goal in this paper is to establish stability of the solutions of the minimum residual problem with respect to these perturbations and approximations. Before we proceed with the theoretical development, however, we discuss an example inverse problem which motivates our studies in this paper.

1.2 An Inverse Problem in Interferometry

Many aerodynamic tests require an evaluation of the flow quality or a determination of flow conditions in the neighborhood of a test article. The intrusive use of probes to measure flow properties can be impractical because of extremely harsh experimental conditions. Even in moderate testing environments, probes can lead to significant flow interference. Also, probes cannot be used to obtain instantaneous distributed information about a dynamic flow field. Naturally, to avoid these problems, nonintrusive measurement systems are preferred when applicable. We seek to develop and analyze numerical techniques to determine distributed flow-field information from nonintrusive optical data.

Interferometry is a particular technique in which fluid density variations may be determined from nonintrusive optical measurements. Specifically, monochromatic laser light is passed through a fluid medium and then added to a reference signal that is uncorrupted by the medium. The density variations in the fluid sample cause changes in the light propagation speed and direction; however, the deflection of light can be neglected in the present context. Therefore, when the light passing through the fluid is combined with the reference light, the two signals combine to form an intensity map showing interference patterns. In inverse interferometry, the density distribution is extracted from the intensity map.

If a perfect-gas flow field is axisymmetric, then the density distribution and the intensity map are related by explicit inversion formulae. With these formulae, it can be shown that the density does not depend continuously upon interferometric measurements. Therefore, methods built on such relations, without the incorporation of modern regularization schemes, have suffered from sensitivity to data error.

Mathematical descriptions of the interferometry process are well understood. The texts of [17] and [13] provide the background material on interferometric measurements and models. The basic equation describing the interferometry process is given by (see, e.g., [17] or [12])

$$\phi = \frac{G}{2} \int [\rho - \rho_\infty] \, d\ell,$$

in which ρ is the flow-field density, ρ_∞ is the free-stream density, and ϕ is the fringe shift of the light. Also, G is a constant depending on the light wavelength and the rate of change of refractive index with respect to density for the specific medium. Finally, $d\ell$ denotes the line integral taken over the path traveled by the light passing through the flow field. Assuming that the flow is axisymmetric and that the light path can be approximated by a straight line, one obtains the simpler relationship

$$\phi(x, y) = G \int_y^R \frac{f(x, r)}{\sqrt{r^2 - y^2}} \, r dr \tag{1.1}$$

where $f(x, r)$ is the axisymmetric representation of the flow-field density perturbation, $\rho - \rho_\infty$. For this expression, it is assumed that $d\ell = dz$, and that the x-axis is coincident with the axis of symmetry. Also, $r = \sqrt{y^2 + z^2}$ is the radial variable and R is the radius of the axisymmetric configuration. The flow-field density perturbation in the region of interest thus determines the phase difference between the light signal passing through the medium and the reference light signal. We denote this functional relationship by $\Phi(f)$.

The goal in inverse interferometry is to determine f based on observed data g. That is, given a particular observed fringe shift pattern, $g(x, y)$, the requirement is to solve the equation $\Phi(f) = g$ for the flow-field density perturbation, $f(x, r)$. After introducing a transformation to make the integral in Eq. (1.1) appear as a convolution, Laplace transforms can be used to obtain the explicit inversion formula,

$$f(x, r) = -\frac{2}{\pi G} \int_r^R \frac{\partial_y g(x, y)}{\sqrt{y^2 - r^2}} \, dy.$$

With the sequence of phase maps $g_n(x, y) = \sin(n^2 y)/n$ that converge to 0 as $n \to \infty$, the above formula gives a corresponding divergent sequence of density perturbations $f_n(x, r)$. Therefore, the mapping from g to f is discontinuous and the inverse problem is ill-posed. The consequence of this crucial fact is that small perturbations in g could potentially translate into large discrepancies in f. This discontinuity property has provided the impetus for a great deal of research aimed at stabilizing the inversion; see [11]. Much of this work focuses primarily on some smoothing of the observed data g for the purposes of differentiation. Here, we take a regularized minimal residual approach to estimating the flow-field density from phase data.

1.3 Residuals and Minimum Norm Fit-to-Data in Interferometry

Minimal residual methods have been used with great success in many ill-posed inverse problems; see [2] and the many references therein. Rather than trying to solve the equation $\Phi(f) = g$ directly via the inversion formula, minimizers of the following cost functional are sought

$$J(f) = \|\Phi(f) - g\|_{L^1(P)}.$$

Here, $g \in L^1(P)$ is the observed fringe shift, i.e., the data, and $\Phi(f)$ denotes the fringe shift corresponding to the density perturbation $f \in L^1(F)$ computed using Eq. (1.1). The intention in the present approach to inverse interferometry is to minimize J over an appropriate collection of density perturbations.

It is a well-known feature of ill-posed inverse problems that to avoid instabilities, such as those discussed above, the set of admissible solutions must be carefully restricted [2]. For example, as a result of the discontinuity property of the inverse map explained earlier, it is possible that for a sequence of density perturbations, $\{f_n\}$, the values, $\{J(f_n)\}$, approach a minimum of J while the densities diverge. Therefore, the set of admissible solutions should be selected to prohibit such divergence. Specifically, the set should be compact with respect to some topology compatible with the cost J.

There are many ways to restrict J to a set of density perturbations which is compact in the larger set of functions in $L^1(F)$. Generally, such a restriction involves a smoothing constraint. However, to be physically acceptible, the set of admissible densities must not rule out density fields with shocks. On the other hand, all flow-field variables are commonly assumed to have bounded total variation. Moreover, as shown in [10], we have that the set

$$\mathcal{S} = \Big\{ f \in L^1(F) \colon |f| \le \alpha, TV(f) \le \beta \Big\} \tag{1.2}$$

is a compact subset of $L^1(F)$.

Implementation of the minimization of J in inverse problems is made very difficult by the need for compactness constraints such as in $\mathcal{S}$. The constraints are not easily incorporated directly into iterative minimization algorithms, so penalty terms are often included in the cost to obtain compactness implicitly. As in our abstract problem above, we are led to a penalized minimal residual cost

$$J_\mu(f) = J(f) + \mu TV(f)$$

in which the penalty term $\mu TV(f)$ is added so that the boundedness of $J_\mu(f)$ enforces the boundedness of $TV(f)$. Note that if μ is chosen sufficiently small, then the perturbed cost functional will be close to the original. Indeed, as $\mu \to 0$, the cost functional, J_μ, will converge uniformly on $\mathcal{S}$ to J. However, when minimizing J_μ, we do not necessarily obtain an element of $\mathcal{S}$, unless at least μ is very large. Thus, we are led to the study of J_μ on an unconstrained set such as $BV(F)$.

1.4 Problems of Approximation

Another crucial issue in the use of penalized residual methods for problems such as interferometry is that of approximation of the images. As $L^1(F)$ is infinite dimensional, one must find appropriate finite dimensional approximation spaces for computational implementation. That is, one must minimize an approximate cost, say J_μ^N, over a finite dimensional subspace X^N of $L^1(F)$. The convergence issue then is to determine if the minimizers of J_μ^N over X^N converge to the minimizers of J_μ over $L^1(F)$.

Such problems have been studied by several authors in other contexts; see, e.g., [16], [2], [14]. Our focus here is on piecewise constant function approximation in $L^1(F)$ with respect to $BV(F)$ penalties. While the use of piecewise constants implies *low order* approximation properties, their use is appropriate since the desire is to capture discontinuities in the images, as observed for transonic flow-field densities. As will be seen below, the use of standard TV norms with piecewise constant functions leads to inconsistent, i.e., non-convergent approximations. We describe below how to correct this problem, and we use these results to build upon the regularization and total variation results of [1], [16], [14].

2 Regularized Minimal Residual Problems

Many researchers have studied identification via penalized minimal residuals, and many results are available for application in particular problems. The interested reader may consult

[2] and the extensive bibliography therein. Rather than review the vast literature, we focus on a few specific results and relate them to the problem at hand.

We begin with the cost functional

$$J_\mu(f) = D(\Phi f - g) + \mu U(f)$$

where $f \in X$ is the quantity to be estimated, $g \in Z$ is the observed data, $\Phi : X \to Z$ is a linear observation operator, $D : Z \to \mathbb{R}$ is a measure of the residual $\Phi f - g$ and $U : Y \subset X \to \mathbb{R}$ is the penalty function. We assume that X, Y, and Z are Banach spaces. For instance, we are interested in the case that $X = L^1(F)$, $Y = BV(F)$, and $Z = L^1(P)$, where F is a bounded flow-field domain in $\mathbb{R}^d$, and P is a bounded measurement domain in $\mathbb{R}^2$.

For the computational setting, we consider a sequence of approximating cost functionals

$$J_\mu^N(f_N) = D^N(\Phi^N f_N - g_N) + \mu U^N(f_N)$$

where $f_N \in X^N$, $g_N \in Z^N$, $\Phi \approx \Phi^N : X^N \to Z^N$, and $D \approx D^N : Z^N \to \mathbb{R}$, for finite dimensional spaces $X^N \subset X$ and $Z^N \subset Z$. Also, it is assumed that $X^N \subset Y$ and $U \approx U^N : X^N \to \mathbb{R}$. We remark here that the motivation for the approximating penalties U^N and residuals D^N is in finding differentiable approximations to the L^1 and TV norms.

2.1 Existence of Minimizers

Here we provide conditions for the existence of minimizers for our cost functionals. We say that a functional $T : Y \to \mathbb{R}$ is Y-coercive if $T(y) \to \infty$ as $|y|_Y \to \infty$. With this definition, we state and prove the following result.

Theorem 2.1 Assume that Y is compactly imbedded into X, and that J_μ and J_μ^N are Y-coercive and lower semicontinuous on X amd X^N, respectively. Then J_μ and J_μ^N have minimizers.

Proof. We only consider the infinite dimensional case. Let $\{f_n\}$ be a minimizing sequence for J_μ. By coercivity, $\{f_n\}$ is Y-bounded. By the compactness of Y in X, there is a subsequence $\{f_{n'}\}$ and an $\bar{f} \in X$ such that $\|f_{n'} - \bar{f}\|_X \to 0$ as $n' \to \infty$. By lower semicontinuity,

$$J_\mu(\bar{f}) \leq \liminf_{n' \to \infty} J_\mu(f_{n'}) = \lim_{n' \to \infty} J_\mu(f_n) = \min J_\mu$$

so $\bar{f}$ is a minimizer. ∎

2.2 Convergence of Minimizers

Now we provide conditions for the convergence of minimizers for J_μ^N to minimizers for J_μ. To obtain such results, we will need some conditions on the way the approximate cost functionals and the approximation spaces relate to their infinite dimensional counterparts. Toward that end, we make the following definitions.

We say that a sequence of functionals $T^N : X^N \to \mathbb{R}$ are uniformly Y-coercive if there is a positive function $F : \mathbb{R} \to \mathbb{R}$ bounded on compact intervals with $\lim_{t\to\infty} F(t) = +\infty$, such that $\forall N$, $T^N(u) \geq F(\|u\|_Y), \forall u \in \mathrm{dom}(T^N)$. Note that this condition is stronger than Y coercivity. We say that the functionals J_μ^N are consistent with J_μ if, given $B > 0$, $\varepsilon > 0$, $\mu_0 > 0$, there is an $\bar{N}$ such that for $N \geq \bar{N}$

$$|J_\mu^N(\chi) - J_\mu(\chi)| < \varepsilon \qquad \forall \mu \in [0, \mu_0], \;\; \forall \chi \in X^N \text{ with } \|\chi\|_Y \leq B.$$

We say that X^N is a J_μ-approximation of X if, for each $f \in X$ with $U(f) < \infty$, there exists a sequence $\{\tilde{f}_N\} \subset X^N$ such that

$$\|f - \tilde{f}_N\|_X \to 0$$

and

$$|J_\mu(f) - J_\mu(\tilde{f}_N)| \to 0, \qquad \forall \mu \geq 0.$$

Before establishing convergence of minimizers, we prove the following useful lemma.

Lemma 2.1 Assume that for each $\mu > 0$, the sequence J_μ^N together with J_μ is uniformly Y-coercive. Also assume that the functionals J_μ^N are consistent with J_μ, and that the X^N spaces are a J_μ-approximation of X. For $f \in X$, let $\{\tilde{f}_N\}$ be chosen to approximate f according to the J_μ-approximation property. Then, given $\mu > 0$, $\varepsilon > 0$, there is an N_0 such that for $N \geq N_0$,

$$J_\mu^N(\tilde{f}_N) \leq J_\mu(f) + \varepsilon.$$

Proof. By uniform coercivity,

$$F_\mu(\|\tilde{f}_N\|_Y) \leq J_\mu(\tilde{f}_N).$$

According to the J_μ-approximation property, there is an $N_1 > 0$ such that for $N \geq N_1$,

$$J_\mu(\tilde{f}_N) \leq J_\mu(f) + \varepsilon/2.$$

The last two inequalities imply $\{\tilde{f}_N\}$ is Y-bounded. By consistency, there is an $N_2 \geq N_1$ such that for $N \geq N_2$,

$$J_\mu^N(\tilde{f}_N) \leq J_\mu(\tilde{f}_N) + \varepsilon/2.$$

Combining the last two inequalities gives the result. ∎

Under conditions given above, we have the following convergence of minimizers.

Theorem 2.2 Assume that the hypotheses of Theorem 2.1 and Lemma 2.1 hold. Fix $\mu > 0$. Suppose f_N^* minimizes J_μ^N over X^N. Then $\{f_N^*\}$ has a subsequence that converges in X, and the limit minimizes J_μ over X. If J_μ has a unique minimizer, say f^*, then $\|f_N^* - f^*\|_X \to 0$ as $N \to \infty$.

Proof. Choose an $\bar{f} \in X$ such that $J_\mu(\bar{f}) < \infty$. Let $\{\bar{f}_N\}$ be chosen to approximate $\bar{f}$ according to the J_μ-approximation property. Given $\varepsilon > 0$, Lemma 2.1 guarantees an $N_0 > 0$ such that for $N \geq N_0$,

$$J_\mu^N(\bar{f}_N) \leq J_\mu(\bar{f}) + \varepsilon.$$

By uniform Y-coercivity and the definition of f_N^*, for $N \geq N_0$,

$$F_\mu(\|f_N^*\|_Y) \leq J_\mu^N(f_N^*) \leq J_\mu^N(\bar{f}_N) \leq J_\mu(\bar{f}) + \varepsilon.$$

Thus, $\{f_N^*\}$ is Y-bounded. By the compactness of Y in X, there is a subsequence $\{f_{N'}^*\}$ and an $\hat{f} \in X$ such that $\|\hat{f} - f_{N'}^*\|_X \to 0$ as $N' \to \infty$. By lower semicontinuity,

$$\begin{aligned} J_\mu(\hat{f}) &\leq \liminf_{N' \to \infty} J_\mu(f_{N'}^*) \\ &\leq \limsup_{N' \to \infty} \left[J_\mu(f_{N'}^*) - J_\mu^{N'}(f_{N'}^*) \right] + \liminf_{N' \to \infty} J_\mu^{N'}(f_{N'}^*) \end{aligned}$$

By consistency, the lim sup is zero. Suppose f^* minimizes J_μ and let $\{\tilde{f}_N\}$ be chosen to approximate f^* according to the J_μ-approximation property. Continuing the last inequality and using the definition of $f_{N'}^*$,

$$J_\mu(\hat{f}) \leq \liminf_{N' \to \infty} J_\mu^{N'}(f_{N'}^*) \leq \liminf_{N' \to \infty} J_\mu^{N'}(\tilde{f}_{N'})$$

Given ε, Lemma 2.1 guarantees an $N_0 > 0$ such that for $N \geq N_0$,

$$J_\mu^N(\tilde{f}_N) \leq J_\mu(f^*) + \varepsilon$$

Combining the last two inequalities gives

$$J_\mu(\hat{f}) \le J_\mu(f^*) + \varepsilon$$

and hence $\hat{f}$ is a minimizer for J_μ. Now suppose f^* is the unique minimizer of J_μ. Suppose $\{f_N^*\}$ has a subsequence $\{f_{N'}^*\}$ such that $\|f^* - f_{N'}^*\|_X \ge \varepsilon > 0$. By the above argument, $\{f_{N'}^*\}$ has a subsequence converging to a minimizer of J_μ. However, this contradicts the uniqueness of f^*. ■

We next establish convergence of minimizers in the limit of increasing data fidelity and vanishing regularization. In order to allow the regularization parameter to go to 0, we must make some more assumptions on the specific ingredients of the cost functionals.

(A1) D and D^N are continuous on Z and Z^N, respectively, with D being uniformly continuous on Z-bounded sets and $D(0) = 0$. Also, given $B > 0$, $\varepsilon > 0$, there is an $\bar{N}$ such that for $N \ge \bar{N}$

$$|D^N(\psi) - D(\psi)| < \varepsilon \qquad \forall \psi \in Z^N \quad \text{with} \quad \|\psi\|_Z \le B$$

Moreover, there exists a constant $c_1 > 0$ such that

$$c_1 \|v\|_Z \le D(v) \quad \forall v \in Z$$

and $\forall N$

$$c_1 \|\psi\|_Z \le D^N(\psi) \quad \forall \psi \in Z^N$$

(A2) U and U^N are lower semicontinuous on X and X^N, respectively. Also, given $B > 0$, $\varepsilon > 0$, there is an $\bar{N}$ such that for for $N \ge \bar{N}$

$$|U^N(\chi) - U(\chi)| < \varepsilon \qquad \forall \chi \in X^N \quad \text{with} \quad \|\chi\|_Y \le B.$$

Moreover, there is a constant $c_2 > 0$ such that

$$c_2 |u|_Y \le U(u) \quad \forall u \in Y$$

and $\forall N$

$$c_2 |\chi|_Y \le U^N(\chi) \quad \forall \chi \in X^N$$

(A3) $\Phi : X \to Z$ and $\Phi^N : X^N \to Z^N$ are bounded linear operators and the sequence $\{\Phi^N\}$ converges strongly to Φ. Also, $\|g - g_N\|_Z \to 0$.

Some remarks are in order at this point. Lower semicontinuity is implied by the assumptions **(A1-3)**. Also, the compactness of Y into X, together with **(A1-3)**, imply consistency, in view of the fact that strong convergence of a sequence of linear operators implies uniform convergence on compact subsets.

Theorem 2.3 Assume that Y is compactly imbedded into X. Assume that for each $\mu > 0$, the sequence J_μ^N together with J_μ is uniformly Y-coercive. Assume that the X^N spaces are a J_μ-approximation of X. Assume that conditions **(A1-3)** hold. Suppose there exists a unique f^* such that $\Phi f^* = g$ and $U(f^*) < \infty$. Let $\{\tilde{f}_N\}$ be chosen to approximate f^* according to the J_μ-approximation property. Suppose $\{\mu_N\} \subset \mathbb{R}^+$ are chosen so that

$$\mu_N \to 0 \qquad \text{and} \qquad \mu_N^{-1} D^N(\Phi^N \tilde{f}_N - g_N) \to 0 \qquad \text{as } N \to \infty. \tag{2.3}$$

Let f_N^* denote a minimizer of $J_{\mu_N}^N$ over X^N. Then $\|f^* - f_N^*\|_X \to 0$ as $N \to \infty$.

Proof. Let $\nu = \max\{1, \{\mu_N\}\}$ and let $\varepsilon > 0$. By Lemma 2.1, there is an $N_0 > 0$ such that for $N \geq N_0$,

$$J_\nu^N(\tilde{f}_N) \leq J_\nu(f^*) + \varepsilon \tag{2.4}$$

By the definition of f_N^*,

$$D^N(\Phi^N f_N^* - g_N) \leq J_{\mu_N}^N(f_N^*) \leq J_{\mu_N}^N(\tilde{f}_N) \leq J_\nu^N(\tilde{f}_N)$$

From the last two inequalities we have that $\{D^N(\Phi^N f_N^* - g_N)\}$ is bounded. Again, by the definition of f_N^*,

$$\begin{aligned} U^N(f_N^*) &\leq \mu_N^{-1} J_{\mu_N}^N(f_N^*) \leq \mu_N^{-1} J_{\mu_N}^N(\tilde{f}_N) \\ &= \mu_N^{-1} D^N(\Phi^N \tilde{f}_N - g_N) + U^N(\tilde{f}_N) \\ &\leq \mu_N^{-1} D^N(\Phi^N \tilde{f}_N - g_N) + J_\nu^N(\tilde{f}_N). \end{aligned}$$

Combining this with Eqs. (2.3) and (2.4), we have that $\{U^N(f_N^*)\}$ is bounded. Thus, $J_\nu(f_N^*)$ is bounded. By uniform coercivity,

$$F_\nu(\|f_N^*\|_Y) \leq J_\nu^N(f_N^*)$$

and hence $\{f_N^*\}$ is Y-bounded. Suppose for the sake of contradiction that there exists a subsequence $\{f_{N'}^*\}$ such that $\|f^* - f_{N'}^*\|_X \geq \varepsilon$ for some $\varepsilon > 0$. By compactness of the imbedding $Y \hookrightarrow X$, there exists a subsequence $\{f_{N''}^*\}$ and an $\hat{f} \in X$ such that $\|\hat{f} - f_{N''}^*\|_X \to 0$ as $N'' \to \infty$. By **(A3)**,

$$\begin{aligned} \|\Phi\hat{f} - g\|_Z &\leq \|\Phi(\hat{f} - f_{N''}^*)\|_Z + \|\Phi f_{N''}^* - g\|_Z \\ &\leq \|\Phi\|_{X,Z} \cdot \|\hat{f} - f_{N''}^*\|_X + \|\Phi f_{N''}^* - g\|_Z \end{aligned}$$

The first term vanishes and we will now show that the second term must also. By Eq. (2.4) and uniform coercivity,

$$F_\nu(\|\tilde{f}_N\|_Y) \leq J_\nu^N(\tilde{f}_N) \leq J_\nu(f^*) + \varepsilon.$$

So $\{\tilde{f}_N\}$ is Y-bounded. Set

$$B = \sup_N \{\{\|f_N^*\|_Y\}, \{\|\tilde{f}_N\|_Y\}\}$$

By consistency, there is an $\bar{N}$ such that for $N \geq \bar{N}$,

$$|J_{\mu_{N''}}(f_{N''}^*) - J_{\mu_{N''}}^{N''}(f_{N''}^*)| \leq \varepsilon/2$$

and

$$|J_{\mu_{N''}}(\tilde{f}_{N''}) - J_{\mu_{N''}}^{N''}(\tilde{f}_{N''})| \leq \varepsilon/2$$

Using the last two inequalities with **(A1)** and the definition of $f_{N''}^*$ gives

$$\begin{aligned} c_1\|\Phi f_{N''}^* - g\|_Z &\leq D(\Phi f_{N''}^* - g) \leq J_{\mu_{N''}}(f_{N''}^*) \leq J_{\mu_{N''}}^{N''}(f_{N''}^*) + \varepsilon/2 \\ &\leq J_{\mu_{N''}}^{N''}(\tilde{f}_{N''}) + \varepsilon/2 \leq J_{\mu_{N''}}(\tilde{f}_{N''}) + \varepsilon \\ &= D(\Phi\tilde{f}_{N''} - g) + \mu_{N''} U(\tilde{f}_{N''}) + \varepsilon \end{aligned}$$

By **(A3)**, $D(\Phi\tilde{f}_{N''} - g)$ converges to $D(\Phi f^* - g)$ and, by the J_μ-approximation property, $U(\tilde{f}_{N''})$ converges to $U(f^*)$. By **(A1)**, $D(\Phi f^* - g) = D(0) = 0$, so the first term vanishes. Since $U(f^*)$ is bounded, the second term also vanishes because of Eq. (2.3). Since ε is arbitrary, $\|\Phi f_{N''}^* - g\|_Z$ vanishes. Hence, $\Phi\hat{f} = g$, but this contradicts the uniqueness of f^*. ∎

We remark here that the compactness assumption, as well as **(A1-3)**, are conditions directly verifiable for particular problems. In many problems, the J_μ approximation property is also straightforward to verify, but in the context of TV regularization it is much more difficult to obtain, as well shall see in the next section. The uniform coercivity condition is often obtained by taking U to be the Y norm. However, when a seminorm is used, the residual part of the cost functional must be used to help provide the coercivity. The following condition, which is similar to the conditions used in [1] and [16], will be used to obtain uniform coercivity.

(A4) There are bounded linear operators $A : X \to \mathcal{N} \subset X$ and $A^N : X^N \to \mathcal{N}^N \subset X^N$ such that

$$\|(I-A)u\|_X + \|(I-A)u\|_Y \le c_3|u|_Y \quad \forall u \in X$$

and

$$\|(I-A^N)\chi\|_X + \|(I-A^N)\chi\|_Y \le c_3|\chi|_Y \quad \forall \chi \in X^N$$

for some constant c_3. Moreover, Φ and Φ^N satisfy

$$\|\Phi A u\|_Z \ge c_4\|Au\|_Y \quad \forall u \in X$$

and

$$\|\Phi^N A^N \chi\|_Z \ge c_4\|A^N\chi\|_Y \quad \forall \chi \in X^N$$

for some constant $c_4 > 0$.

Proposition 2.1 Assume **(A1)** through **(A4)**. Then, for each $\mu > 0$, the sequence J_μ^N together with J_μ is uniformly Y-coercive.

Proof. We only consider the inequality for the finite dimensional spaces. The inequality for the infinite dimensional setting is established simlarly. First note that

$$\begin{aligned}
\left[J_\mu^N(\chi)\right]^2 &\ge \left[D^N(\Phi^N\chi - g_N)\right]^2 + \left[\mu U^N(\chi)\right]^2 \\
&\ge c_1^2\|\Phi^N\chi - g_N\|_Z^2 + c_2^2\mu^2|\chi|_Y^2 \\
&\ge c_1^2\|\Phi^N A^N\chi\|_Z\left(\|\Phi A^N\chi\|_Z - 2\|\Phi^N(I-A^N)\chi - g_N\|_Z\right) + \mu^2c_2^2|\chi|_Y^2
\end{aligned}$$

As a result,

$$J_\mu^N(\chi) \ge \mu c_2|\chi|_Y \tag{2.5}$$

By **(A3)**, $G = \sup_N\{\|g_N\|_Z\}$ is finite. Also, by **(A3)** and the Principle of Uniform Boundedness, $C = \sup_N\{\|\Phi^N\|_{\mathcal{L}(X,Z)}\}$ is finite. Then, with **(A4)** we see that

$$\begin{aligned}
\|\Phi^N(I-A^N)\chi - g_N\|_Z &\le C\|(I-A^N)\chi\|_X + \|g_N\|_Z \\
&\le Cc_3|\chi|_Y + G
\end{aligned}$$

Combining the above, we have that

$$\left[J_\mu^N(\chi)\right]^2 \ge c_1^2c_4\|A^N\chi\|_Y\left[c_4\|A^N\chi\|_Y - 2\left(Cc_3|\chi|_Y + G\right)\right] + \mu^2c_2^2|\chi|_Y^2$$

If the bracketed expression is greater than 1, then

$$\left[J_\mu^N(\chi)\right]^2 \ge c_1^2c_4\|A^N\chi\|_Y$$

Using this with **(A4)** and Eq. (2.5) gives

$$\begin{aligned}\|\chi\|_Y &\leq \|A^N\chi\|_Y + \|(I - A^N)\chi\|_Y \leq c_1^{-2}c_4^{-1}\left[J_\mu^N(\chi)\right]^2 + c_3|\chi|_Y \\ &\leq c_1^{-2}c_4^{-1}\left[J_\mu^N(\chi)\right]^2 + c_3c_2^{-1}\mu^{-1}J_\mu(\chi).\end{aligned}$$

This establishes uniform Y-coercivity since none of these constants depend upon N. On the other hand, if the bracketed expression above is bounded by 1, then using **(A4)** and Eq. (2.5),

$$\begin{aligned}\|\chi\|_Y &\leq \|A^N\chi\|_Y + \|(I - A^N)\chi\|_Y \\ &\leq c_4^{-1}(1 + 2Cc_3|\chi|_Y + 2G) + c_3|\chi|_Y \\ &\leq c_4^{-1}(1 + 2G) + \left(1 + 2Cc_4^{-1}\right)c_3c_2^{-1}\mu^{-1}J_\mu^N(\chi).\end{aligned}$$

Again, uniform Y-coercivity is established since none of these constants depend upon N. ■

As noted above, the J_μ-approximation property is surprisingly difficult to analyze in the context of total variation penalty terms with piecewise constant approximating elements, and in the next section we study this issue in-depth. For the case of continuous, piecewise linear elements, the following simple result provides the verification of the J_μ-approximation property.

Proposition 2.2 Assume **(A1-4)**. Suppose that $F \subset \mathbb{R}^d$, that $X = L^1(F)$, and that $U(f) = TV(f)$. Suppose that $\{X^N\}$ is a sequence of spaces of piecewise linear continuous functions with the property that

$$\inf_{\chi \in X^N} \|f - \chi\|_{W^{1,1}(F)} \to 0 \qquad \text{as} \qquad N \to \infty \qquad\qquad \forall f \in W^{1,1}(F).$$

Then the J_μ-approximation property holds.

Proof. Let $f \in X$ with $U(f) < \infty$ so $f \in BV(F)$. Fix $\mu \geq 0$. According to Theorem 2, p. 172 of [7], there exist functions $\{\phi_M\} \subset C^\infty(F) \cap BV(F)$ such that

$$\|f - \phi_M\|_{L^1(F)} \leq \frac{1}{2M}$$

and

$$|TV(f) - TV(\phi_M)| \leq \frac{1}{2M}.$$

Then for each M, choose $N(M) > N(M-1)$ so that for $N' \geq N(M)$, there exists a $\chi_{MN'} \in X^{N'}$ such that

$$\|\phi_M - \chi_{MN'}\|_{W^{1,1}(F)} \leq \frac{1}{2M}.$$

Given any N define $M(N)$ as that M for which $N(M) \leq N < N(M+1)$. Such an M always exists since $N(M)$ is strictly increasing without bound. Also, since $N(M)$ is never infinite for any finite M, $M(N) \to \infty$ as $N \to \infty$. Now define $f_N = \chi_{M(N)N}$ to get

$$\|\phi_{M(N)} - f_N\|_{L^1(F)} + TV(\phi_{M(N)} - f_N) = \|\phi_{M(N)} - f_N\|_{W^{1,1}(F)} \leq \frac{1}{2M(N)}.$$

Hence,

$$\|f - f_N\|_{L^1(F)} \leq \|f - \phi_{M(N)}\|_{L^1(F)} + \|\phi_{M(N)} - f_N\|_{L^1(F)} \leq \frac{1}{M(N)} \to 0. \tag{2.6}$$

and

$$|TV(f) - TV(f_N)| \le |TV(f) - TV(\phi_{M(N)})| + |TV(\phi_{M(N)} - f_N)| \le \frac{1}{M(N)} \to 0. \qquad (2.7)$$

Now let $\varepsilon > 0$. By **(A3)**, and Eq. (2.6), the following is finite,

$$B = \max\{\|\Phi f - g\|_Z, \sup_N\{\|\Phi f_N - g\|_Z\}\}.$$

According to **(A7)** we can choose $\delta > 0$ so that

$$|D(v_2) - D(v_1)| < \varepsilon/2 \qquad \text{for} \qquad \|v_2 - v_1\|_Z < \delta, \quad \|v_1\|_Z, \|v_2\|_Z \le B.$$

Then using **(A9)** and Eq. (2.6), we can choose $\bar{N}$ such that for $N \ge \bar{N}$

$$\|(\Phi f - g) - (\Phi f_N - g)\|_Z = \|\Phi(f - f_N)\|_Z \le \|\Phi\|_{X,Z}\|f - f_N\|_X < \delta.$$

Also, by Eq. (2.7), assume $\bar{N}$ is large enough that for $N \ge \bar{N}$,

$$\mu|U(f) - U(f_N)| < \varepsilon/2.$$

Combining the above gives that for $N \ge \bar{N}$

$$|J_\mu(f) - J_\mu(f_N)| \le |D(\Phi f - g) - D(\Phi f_N - g)| + \mu|U(f) - U(f_N)| \le \varepsilon$$

which establishes the J_μ-approximation property. ∎

Thus, for linear spline approximation, we have verified the J_μ-approximation property. However, piecewise constant functions are commonly used in these problems, and it is of interest to study the J_μ-approximation property in this context. This is the purpose of the next section.

3 Piecewise Constant Approximation in BV

An important issue in implementation is that of finite dimensional approximation. To approximate discontinuous functions, piecewise constant functions on a fixed grid are a natural choice of approximating elements. We shall see, however, that care must be taken to ensure that in some sense the approximate problems converge to the original problem. In this section we give an example and propose some alternative approximation approaches. We remark here that similar results have been derived in [4] in the context of image reconstruction.

3.1 A Simple Example

Here we present an example of a function f approximated by piecewise constant functions f_N in such a way that $f_N \to f$ uniformly, but $TV(f_N)$ does not converge to $TV(f)$. Let F be discretized with a smooth curvilinear grid defined by cell vertices, $\{\boldsymbol{x}_{i+\frac{1}{2},j+\frac{1}{2}}\}$, $0 \le i,j \le N$, and cell centers, $\{\boldsymbol{x}_{ij}\}$, $1 \le i,j \le N$, where $\boldsymbol{x} = (x,r)$. Let the grid be parametrized by $\boldsymbol{\xi}(\boldsymbol{x})$ where, given a $\Delta\boldsymbol{x}$ along a grid curve, the corresponding $\Delta\boldsymbol{\xi}$ gives the length of the grid curve segment. Specifically, with $\boldsymbol{\xi}_{ij} = \boldsymbol{\xi}(\boldsymbol{x}_{ij})$ and $\boldsymbol{\xi} = (\xi,\eta)$, the lengths of grid curves from $\boldsymbol{x}_{i-\frac{1}{2},j-\frac{1}{2}}$ to $\boldsymbol{x}_{i+\frac{1}{2},j-\frac{1}{2}}$ and from $\boldsymbol{x}_{i-\frac{1}{2},j-\frac{1}{2}}$ to $\boldsymbol{x}_{i-\frac{1}{2},j+\frac{1}{2}}$ are given by $\Delta_i\xi_{i-\frac{1}{2},j-\frac{1}{2}}$ and $\Delta_j\eta_{i-\frac{1}{2},j-\frac{1}{2}}$, where Δ_i and Δ_j denote forward difference operators with respect to i and j. Now, let $f \in C^1(F)$ so that $TV(f) = \int_F |\nabla f|_{\ell_2}$. Define f_N by

$$f_N(\boldsymbol{x}) = \sum_{i,j=1}^{N} f(\boldsymbol{x}_{ij})\chi_{C_{ij}}(\boldsymbol{x}) \qquad \text{with} \quad \chi_{C_{ij}}(\boldsymbol{x}) = \begin{cases} 1 & \xi_{i-\frac{1}{2},j} \le \xi(\boldsymbol{x}) < \xi_{i+\frac{1}{2},j}, \\ & \eta_{i,j-\frac{1}{2}} \le \eta(\boldsymbol{x}) < \eta_{i,j+\frac{1}{2}} \\ 0 & \text{otherwise} \end{cases}$$

To compute $TV(f_N)$ we use Theorem 1, p. 183 of [7]. Specifically, let C be a generic cell from the grid and let χ_C denote the indicator or characteristic function of C. Then for $\chi_1 = w_1\chi_C$ and $\chi_2 = w_2[1-\chi_C]$, the function $\chi = \chi_1 + \chi_2$ satisfies

$$TV_F(\chi) = TV_C(\chi_1) + TV_{F-C}(\chi_2) + \int_{\partial F} |\chi_1 - \chi_2|$$

where the last term is the $L^1(\partial C)$ norm of the difference of the traces. Thus,

$$TV_F(\chi) = |w_1 - w_2|\mathcal{H}(\partial C)$$

where $\mathcal{H}(\partial C)$ is the Hausdorff measure of the boundary of C, i.e., its length. Continuing in this way, we find that

$$\begin{aligned}
TV(f_N) &= \sum_{i,j=1}^{N-1} \left[|\Delta_i f_{ij}|\Delta_j \xi_{i+\frac{1}{2},j-\frac{1}{2}} + |\Delta_j f_{ij}|\Delta_i \eta_{i-\frac{1}{2},j+\frac{1}{2}}\right] \\
&\quad + \sum_{i=1}^{N-1} \left[|\Delta_i f_{ij}|\Delta_j \xi_{i+\frac{1}{2},j-\frac{1}{2}}\right]_{j=N} + \sum_{j=1}^{N-1} \left[|\Delta_j f_{ij}|\Delta_i \eta_{i-\frac{1}{2},j+\frac{1}{2}}\right]_{i=N} \\
&= \sum_{i,j=1}^{N-1} \left[\frac{|\Delta_i f_{ij}|}{\Delta_i \xi_{i-\frac{1}{2},j+\frac{1}{2}}} + \frac{|\Delta_j f_{ij}|}{\Delta_j \eta_{i+\frac{1}{2},j-\frac{1}{2}}}\right] \Delta_i \xi_{i-\frac{1}{2},j+\frac{1}{2}} \Delta_j \eta_{i+\frac{1}{2},j-\frac{1}{2}} \\
&\quad + \sum_{i=1}^{N-1} \left[|\Delta_i f_{ij}|\Delta_j \xi_{i+\frac{1}{2},j-\frac{1}{2}}\right]_{j=N} + \sum_{j=1}^{N-1} \left[|\Delta_j f_{ij}|\Delta_i \eta_{i-\frac{1}{2},j+\frac{1}{2}}\right]_{i=N}.
\end{aligned}$$

With h denoting the maximum diameter of any cell, we find on passing to the limit that

$$TV(f_N) \stackrel{h\to 0}{\longrightarrow} \int_{\boldsymbol{\xi}(F)} \left[\left|\frac{\partial f}{\partial \xi}\right| + \left|\frac{\partial f}{\partial \eta}\right|\right] d\xi d\eta = \int_F \left|(J^{-1})^T \nabla f\right|_{\ell_1} |J| dx dr$$

where J is the metric Jacobian,

$$J = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial r} \\[2ex] \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial r} \end{bmatrix} \qquad \text{with} \qquad (J^{-1})^T = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial r}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial r}{\partial \eta} \end{bmatrix}.$$

In particular, if the grid is Cartesian, $J = I$ and

$$TV(f_N) \stackrel{h\to 0}{\longrightarrow} \int_F |\nabla f|_{\ell_1} dx dr \neq \int_F |\nabla f|_{\ell_2} dx dr = TV(f).$$

On the other hand, if the grid conforms to the level curves of f with

$$\nabla \xi = \frac{(f_x, f_r)}{\sqrt{f_x^2 + f_r^2}} \qquad \text{and} \qquad \nabla \eta = \frac{(-f_r, f_x)}{\sqrt{f_x^2 + f_r^2}}$$

then $|J| = 1$ and $\nabla f = |\nabla f|_{\ell_2} \nabla \xi$. Therefore,

$$(J^{-1})^T \nabla f = |\nabla f|_{\ell_2} (J^{-1})^T \nabla \xi = |\nabla f|_{\ell_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$|(J^{-1})^T \nabla f|_{\ell_1} = |\nabla f|_{\ell_2}.$$

Hence, $TV(f_N)$ does converge to $TV(f)$ for the right grid. The fact that $TV(f_N)$ does not converge to $TV(f)$ on a realistic grid is a serious problem for the identification problems of interest in this paper. We are specifically interested in using Cartesian grids and the above example indicates that the J_μ-approximation property does not hold in this case. However, an alternative approach is suggested by this example.

3.2 Alternative Definitions of the Total Variation

To obtain the appropriate convergence results, one possibility is to redefine the TV norm in terms of the ℓ_1 norm in $\mathbb{R}^d$. We set

$$TV_p(f) = \sup_{\Psi \in \mathcal{C}_p} \int_F f \nabla \cdot \Psi,$$

where, with $1/q = 1 - 1/p$,

$$\mathcal{C}_p = \{\Psi: F \to \mathbb{R}^d, \Psi \in C_0^1, |\Psi|_{\ell_q} \leq 1\}$$

Note that the usual TV is TV_2. We also note that if $f \in C^1(\bar{F})$, then we have by Hölder's inequality,

$$\int_F f \nabla \cdot \Psi = \int_F \nabla f \cdot \Psi \leq \int_F |\nabla f|_{\ell_p} \qquad \text{for } |\Psi|_{\ell_q} \leq 1.$$

A limiting argument similar to that given in the standard case yields $TV_p(f) = \int_F |\nabla f|_{\ell_p}$ for all $f \in C^1(\bar{F})$.

Note that $TV_p(\chi)$ for a piecewise constant χ is independent of p provided the grid has the Cartesian form $\{(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (r_{j-\frac{1}{2}}, r_{j+\frac{1}{2}})\}$, $1 \leq i, j \leq N$. Specifically, let C be a generic cell from a Cartesian grid and let χ_C denote the characteristic function of C. Then

$$TV_p(\chi_C) = \sup_{\Psi \in \mathcal{C}_p} \int_F \chi_C \nabla \cdot \Psi = \sup_{\Psi \in \mathcal{C}_p} \int_{\partial C} \Psi \cdot \boldsymbol{\nu} dH \leq \sup_{\Psi \in \mathcal{C}_p} \int_{\partial C} |\Psi|_{\ell_q} |\boldsymbol{\nu}|_{\ell_p} dH \leq H(\partial C).$$

Here, H denotes the Hausdorff measure and $\boldsymbol{\nu}$ is an outwardly directed vector with $|\boldsymbol{\nu}|_{\ell_2} = 1$. However, since the grid is Cartesian, $\boldsymbol{\nu}$ has only a single nonzero component necessarily equal to 1. Thus, $|\boldsymbol{\nu}|_{\ell_p} = 1$ for all $p \geq 1$. Again, a limiting argument shows that $TV_p(\chi_C) = H(\partial C)$. Continuing in this way, we find that $TV_p(\chi)$ is independent of p for any piecewise constant χ.

Since we are interested in using Cartesian grids, the above results suggest that to obtain appropriate limiting behavior we should use TV_1. However, we still need many of the well known TV_2 properties to hold for TV_1, including compactness properties. Many of these TV_2 results also hold for TV_1 due to the equivalence of the ℓ_1 and ℓ_2 norms in $\mathbb{R}^d$. In particular, the set of functions of bounded variation is independent of the choice of p. Note that TV_1 has been employed previously [5] to analyze conservation laws, with compactness in L^1 obtained through bounded TV_1 sets. By using TV_1, we obtain the following result which allows us to verify the J_μ-approximation property for piecewise constant functions.

Proposition 3.1 Assume **(A1-4)**. Suppose that F is a rectangle in $\mathbb{R}^2$, that $X = L^1(F)$, and that $U(f) = TV_1(f)$. Suppose $\{X^N\}$ is a sequence of spaces of piecewise constant functions defined on Cartesian grids made of identical cells with vanishing diameter as $N \to \infty$. Then the J_μ-approximation property holds.

Proof. Let $f \in X$ with $U(f) < \infty$ so $f \in BV(F)$. Fix $\mu \geq 0$. According to Theorem 2, p. 172 of [7], f can be approximated by smooth functions $\{\phi_M\}$ in the sense that as $M \to \infty$, $f - \phi_M$ vanishes in $L^1(F)$ while $TV_2(\phi_M)$ converges to $TV_2(f)$. This result holds for TV_1 as well and the extension is nearly identical to that given in [7]. After the prescribed changes to the theorem of [7], choose $\{\phi_M\} \subset C^\infty(F) \cap BV(F)$ such that

$$\|f - \phi_M\|_{L^1(F)} \leq \frac{1}{2M}$$

and

$$|TV_1(f) - TV_1(\phi_M)| \leq \frac{1}{2M}.$$

Then, following the example of the previous subsection, define $\chi_{MN} \in X^N$ by

$$\chi_{MN}(\boldsymbol{x}) = \sum_{i,j=1}^{N} \phi_M(\boldsymbol{x}_{ij})\chi_{C_{ij}}(\boldsymbol{x})$$

where $\chi_{C_{ij}}$ is the characteristic function for the cell $C_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [r_{j-\frac{1}{2}}, r_{j+\frac{1}{2}}]$. As before, we find on passing to the limit that

$$TV_1(\chi_{MN}) \overset{N\to\infty}{\longrightarrow} \int_F |\nabla \phi_M|_{\ell_1} dxdr = TV_1(\phi_M).$$

Also, we note that

$$\|\phi_M - \chi_{MN}\|_{L^1(F)} \overset{N\to\infty}{\longrightarrow} 0.$$

Thus, for each M, we can choose $N(M) > N(M-1)$ so that for $N' \geq N(M)$, there exists a $\chi_{MN'} \in X^{N'}$ such that

$$\|\phi_M - \chi_{MN'}\|_{L^1(F)} \leq \frac{1}{2M}$$

and

$$|TV_1(\phi_M) - TV_1(\chi_{MN'})| \leq \frac{1}{2M}.$$

Given any N define $M(N)$ as that M for which $N(M) \leq N < N(M+1)$. Such an M always exists since $N(M)$ is strictly increasing without bound. Also, since $N(M)$ is never infinite for any finite M, $M(N) \to \infty$ as $N \to \infty$. Now define $f_N = \chi_{M(N)N}$ to get

$$\|\phi_{M(N)} - f_N\|_{L^1(F)} \leq \frac{1}{2M(N)}$$

and

$$\left|TV_1(\phi_{M(N)}) - TV_1(f_N)\right| \leq \frac{1}{2M(N)}.$$

Hence,

$$\|f - f_N\|_{L^1(F)} \leq \|f - \phi_{M(N)}\|_{L^1(F)} + \|\phi_{M(N)} - f_N\|_{L^1(F)} \leq \frac{1}{M(N)} \to 0$$

and

$$|TV_1(f) - TV_1(f_N)| \leq \left|TV_1(f) - TV_1(\phi_{M(N)})\right| + \left|TV_1(\phi_{M(N)}) - TV_1(f_N)\right| \leq \frac{1}{M(N)} \to 0.$$

As in the proof of Proposition 2.2, the convergence of the residual term is straightforward, and we omit the details here. ■

4 Total Variation Regularization

In this section, we apply our results to particular regularized minimal residual problems. We begin by considering the cost functional

$$J_\mu(f) = \|\Phi f - g\|_Z + \mu TV_1(f). \tag{4.8}$$

Here, the flow-field domain on which the density is supported is

$$F = \{(x, r) : x_{\min} \leq x \leq x_{\max}, 0 \leq r \leq R\}$$

and the measurement domain on which the data is supported is

$$P = \{(x, y) : x_{\min} \leq x \leq x_{\max}, 0 \leq y \leq R\}.$$

Also, $X = L^1(F)$ and $Z = L^1(P)$. The forward operator, Φ, is the interferometry operator of Eq. (1.1). Take $Y = BV(F)$ equipped with the seminorm $|\cdot|_Y = TV_1(\cdot)$ and norm $\|\cdot\|_Y = \|\cdot\|_{L^1(F)} + TV_1(\cdot)$. Let A be the average value operator, $Au = \int_F u/[R(x_{\max} - x_{\min})]$. By inspection, $D(v) = \|v\|_Z$ and $U(u) = TV_1(u)$.

We also consider the approximating cost functionals

$$J_\mu^N(\chi) = \|\Phi^N \chi - g_N\|_Z + \mu TV_1(\chi) \tag{4.9}$$

in addition to others specified later in this section. Here, X^N is a space of piecewise constant functions defined on the Cartesian grid with cell centers at

$$x_i = x_{\min} + (i - \tfrac{1}{2})h_x, \quad r_j = (j - \tfrac{1}{2})h_r, \qquad 1 \le i, j \le N,$$

where $h_x = (x_{\max} - x_{\min})/N$ and $h_r = R/N$. Similarly, Z^N is a space of piecewise constant functions defined on the Cartesian grid with cell centers at

$$x_i = x_{\min} + (i - \tfrac{1}{2})h_x, \quad y_j = (j - \tfrac{1}{2})h_y, \qquad 1 \le i, j \le N,$$

where $h_y = R/N$. Let Π_{Z^N} be an operator which, in rough terms, projects Z into Z_N,

$$\Pi_{Z^N} v = \sum_{i,j=1}^{N} \frac{\chi_{C_{ij}}}{h_x h_y} \int_{C_{ij}} v(x,y)\, dxdy.$$

Here C_{ij} is the cell $C_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$. The discrete forward operator is $\Phi^N = \Pi_{Z_N}\Phi$ and the discrete data is $g_N = \Pi_{Z_N} g$. Let $A^N = A$. By inspection, $D^N(\psi) = \|\psi\|_Z$ and $U^N(\chi) = TV_1(\chi)$.

We now consider these particular choices in relation to assumptions made earlier. First, we consider the compact imbedding condition. Since the TV_1 and TV_2 seminorms are equivalent, the compact imbedding follows from Theorem 4, p. 176 of [7]. Our next concern is the verification of **(A1-4)**. First, note that $\Pi_{Z^N} : Z \to Z^N$ is a bounded linear operator, satisfying

$$\|\Pi_{Z^N} v\|_Z \le \|v\|_Z$$

for all $v \in Z$, with equality holding in case $v \ge 0$. Also, we have that for every $v \in L^1(P)$,

$$\|[I - \Pi_{Z^N}]v\|_{L^1(P)} \to 0 \quad \text{as} \quad N \to \infty.$$

These facts imply that **(A1-3)** hold. The condition **(A4)** is obtained via Poincare's inequality, together with the following computation. Noting that $\Phi(1) = G\sqrt{R^2 - y^2}$, we have $\|\Phi A u\|_Z = \frac{1}{4}|Au|\pi R^2 G(x_{\max} - x_{\min})$. Also, $\|\Pi_{Z^N}\psi\|_Z = \|\psi\|_Z$ for $\psi \ge 0$, so $\|\Phi^N A^N \chi\|_Z = \|\Phi A \chi\|_Z = \frac{1}{4}|A\chi|\pi R^2 G(x_{\max} - x_{\min})$.

From the verification of these conditions, we have the following convergence result.

Theorem 4.1 J_μ and J_μ^N in Eqs. (4.8) and (4.9) have minimizers in $L^1(F)$ and X^N, respectively. Let f_N^* denote a minimizer of J_μ^N over X^N. Then $\{f_N^*\}$ has a subsequence that converges in $L^1(F)$, and the limit minimizes J_μ over $L^1(F)$. Also, if there exists a unique $f^* \in BV(F)$ such that $\Phi f^* = g$, we can choose $\{\mu_N\} \subset \mathbb{R}^+$ to converge to 0 sufficiently slowly so that $\|f^* - f_N^*\|_{L^1(F)} \to 0$ as $N \to \infty$.

To address the uniqueness issue, we have the following lemma.

Lemma 4.1 Assume $\Phi f_1 = \Phi f_2$. Suppose that for each fixed x,

$$F_1(\xi) = \begin{cases} f_1(x, \sqrt{R^2 - \xi}) & 0 \le \xi \le R^2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad F_2(\xi) = \begin{cases} f_2(x, \sqrt{R^2 - \xi}) & 0 \le \xi \le R^2 \\ 0 & \text{otherwise} \end{cases}$$

have Laplace Transforms and are uniquely recoverable from their respective transforms. Then $f_1 = f_2$.

Proof. Since $\Phi[f_2 - f_1] = 0$, it follows with the change of variables, $\xi = R^2 - r^2$, $\eta = R^2 - y^2$, that

$$\int_0^\infty K(\eta - \xi)[F_2(\xi) - F_1(\xi)]d\xi = G(\eta) \tag{4.10}$$

where

$$K(\eta) = \begin{cases} 1/\sqrt{\eta} & \eta > 0 \\ 0 & \text{otherwise} \end{cases}$$

and where G is some function satisfying $G(\eta) = 0$ for $\eta \leq R^2$. Since the left side of Eq. (4.10) has a Laplace Transform, so does the right side. So, the convolution equation in (4.10) can be expressed as

$$\hat{K}(\alpha)[\hat{F}_2(\alpha) - \hat{F}_1(\alpha)] = \hat{G}(\alpha) \tag{4.11}$$

where "ˆ" denotes the Laplace Transform of the underlying function. Since the left side of Eq. (4.10) is uniquely recoverable from the left side of Eq. (4.11), so is $G(\eta)$ uniquely recoverable from $\hat{G}(\alpha)$. Using $\hat{K}(\alpha) = \sqrt{\pi/\alpha}$ in Eq. (4.11) gives

$$\hat{F}_2(\alpha) - \hat{F}_1(\alpha) = \sqrt{\frac{\alpha}{\pi}}\hat{G}(\alpha).$$

Now let $S_\varepsilon(\eta)$ be the usual exponential mollifier which vanishes for $|\eta| > \varepsilon$. Then multiplying the last equation by $\hat{S}_\varepsilon(\alpha)$ leads to

$$\hat{S}_\varepsilon(\alpha)[\hat{F}_2(\alpha) - \hat{F}_1(\alpha)] = \frac{1}{\pi}\sqrt{\frac{\pi}{\alpha}}\left\{\alpha\left[\hat{S}_\varepsilon(\alpha)\hat{G}(\alpha)\right]\right\}.$$

Since $G(\eta)$ is uniquely recoverable from $\hat{G}(\alpha)$, $S_\varepsilon(\eta) \star G(\eta)$ is uniquely recoverable from $\hat{S}_\varepsilon(\alpha)\hat{G}(\alpha)$. Also, since $S_\varepsilon(\eta)\star G(\eta)$ is smooth, $\frac{d}{d\eta}[S_\varepsilon(\eta)\star G(\eta)]$ is the inverse Laplace Transform of $\alpha[\hat{S}_\varepsilon(\alpha)\hat{G}(\alpha)]$. Continuing in this way, we find that

$$S_\varepsilon(\eta) \star [F_2(\eta) - F_1(\eta)] = \frac{1}{\pi}K(\eta) \star \left\{\frac{d}{d\eta}\left[S_\varepsilon(\eta) \star G(\eta)\right]\right\}.$$

Writing these convolutions explicitly gives

$$\int_0^\infty S_\varepsilon(\eta - \xi)[F_2(\xi) - F_1(\xi)]d\xi = \frac{1}{\pi}\int_0^\infty K(\eta - \xi)\left[\frac{d}{d\xi}\int_0^\infty S_\varepsilon(\xi - \zeta)G(\zeta)d\zeta\right]d\xi. \tag{4.12}$$

Since $G(\zeta) = 0$ for $\zeta < R^2$,

$$\frac{d}{d\xi}\int_0^\infty S_\varepsilon(\xi - \zeta)G(\zeta)d\zeta = \int_0^\infty S_\varepsilon(\xi - \zeta)G(\zeta)d\zeta = 0 \quad \text{for} \quad \xi < R^2 - 2\varepsilon.$$

Thus,

$$\int_0^\infty K(\eta - \xi)\left[\frac{d}{d\xi}\int_0^\infty S_\varepsilon(\xi - \zeta)G(\zeta)d\zeta\right]d\xi = 0 \quad \text{for} \quad \eta < R^2 - 2\varepsilon.$$

By Theorem 1, p. 122 of [7], the left side of Eq. (4.12) converges in $L^1([0, R^2])$ to $F_2(\eta) - F_1(\eta)$ as $\varepsilon \to 0$. Since the left side of Eq. (4.12) converges in $L^1([0, R^2])$ so does the right side. Moreover, the right side of Eq. (4.12) converges pointwise to 0 on $[0, R^2)$ as $\varepsilon \to 0$. Thus, $f_1 = f_2$. ■

The nondifferentiability of J_μ leads us to consider the smoothed cost functional

$$J_{\mu,\gamma}(f) = \int_P \sqrt{\gamma^2 + |\Phi f - g|^2}dxdy + \mu\int_F \left[\sqrt{\gamma^2 + |f_x|^2} + \sqrt{\gamma^2 + |f_r|^2}\right]dxdr \tag{4.13}$$

which *rounds off the corner* of the absolute values appearing in J_μ of Eq. (4.8). A similar approach is taken in [1] where $TV_2(f)$ is approximated with $\int[\gamma^2 + |\nabla f|^2_{\ell_2}]^{\frac{1}{2}}$. As shown in

[1] and [9], one may develop a variational formulation of the above penalty which allows its extension to BV functions. Rather than focusing on this issue, we proceed to the construction of finite dimensional approximations for Eq. (4.8) based upon Eq. (4.13).

Because of the nature of convenient minimization methods, we now consider the approximating cost functionals

$$\begin{aligned} J_\mu^N(\chi) &= \sum_{i,j=1}^{N} h_x h_y \sqrt{\gamma_N^2 + |\Phi^N \chi - g_N|_{ij}^2} \\ &+ \mu \sum_{i,j=1}^{N-1} h_x h_r \left[\sqrt{\gamma_N^2 + h_x^{-2} |\Delta_i \chi_{ij}|^2} + \sqrt{\gamma_N^2 + h_r^{-2} |\Delta_j \chi_{ij}|^2} \right] \\ &+ \mu \sum_{i=1}^{N-1} h_x h_r \sqrt{\gamma_N^2 + h_x^{-2} |\Delta_i \chi_{iN}|^2} \ + \ \mu \sum_{j=1}^{N-1} h_x h_r \sqrt{\gamma_N^2 + h_r^{-2} |\Delta_j \chi_{Nj}|^2} \end{aligned} \tag{4.14}$$

All definitions made for the approximating cost functionals in Eq. (4.9) are used here except that now for $\psi \in Z^N$,

$$D^N(\psi) = \sum_{i,j=1}^{N} h_x h_y \sqrt{\gamma_N^2 + |\psi|_{ij}^2}$$

and for $\chi \in X^N$,

$$\begin{aligned} U^N(\chi) &= \sum_{i,j=1}^{N-1} h_x h_r \left[\sqrt{\gamma_N^2 + h_x^{-2} |\Delta_i \chi_{ij}|^2} + \sqrt{\gamma_N^2 + h_r^{-2} |\Delta_j \chi_{ij}|^2} \right] \\ &+ \sum_{i=1}^{N-1} h_x h_r \sqrt{\gamma_N^2 + h_x^{-2} |\Delta_i \chi_{iN}|^2} \ + \ \sum_{j=1}^{N-1} h_x h_r \sqrt{\gamma_N^2 + h_r^{-2} |\Delta_j \chi_{Nj}|^2} \end{aligned}$$

where $\gamma_N \to 0$ as $N \to \infty$.

We now consider these new choices in relation to assumptions made earlier. Because of the similarities between Eq. (4.9) and Eq. (4.14), only **(A1)** and **(A2)** require attention. For these note that since

$$|x| \le \sqrt{\gamma^2 + |x|^2}$$

for $\psi \in Z^N$,

$$D(\psi) \ = \ \|\psi\|_{L^1(P)} \ = \ \sum_{i,j=1}^{N} h_x h_y |\psi| \ \le \ D^N(\psi)$$

and for $\chi \in X^N$,

$$\begin{aligned} U(\chi) \ = \ TV_1(\chi) \ &= \ \sum_{i,j=1}^{N-1} h_x h_r \left[h_x^{-1} |\Delta_i \chi_{ij}| + h_r^{-1} |\Delta_j \chi_{ij}| \right] \\ &+ \sum_{i=1}^{N-1} h_x h_r \left[h_x^{-1} |\Delta_i \chi_{iN}| \right] \ + \ \sum_{j=1}^{N-1} h_x h_r \left[h_r^{-1} |\Delta_j \chi_{Nj}| \right] \ \le \ U^N(\chi) \end{aligned}$$

Also, the inequality,

$$\left| \sqrt{\gamma^2 + |x|^2} - |x| \right| \le \gamma$$

implies

$$|D(\psi) - D^N(\psi)| \le \gamma_N R(x_{\max} - x_{\min}) \to 0 \qquad \text{as} \qquad N \to \infty$$

and

$$|U(\chi) - U^N(\chi)| \leq 2\gamma_N R(x_{\max} - x_{\min}) \to 0 \qquad \text{as} \qquad N \to \infty.$$

Thus, **(A1)** and **(A2)** follow with $c_2 = c_1 = 1$.

From the verification of these conditions, we have the following convergence result.

Theorem 4.2 J_μ and J_μ^N in Eqs. (4.8) and (4.14) have minimizers in $L^1(F)$ and X^N, respectively. Let f_N^* denote a minimizer of J_μ^N over X^N. Then $\{f_N^*\}$ has a subsequence that converges in $L^1(F)$, and the limit minimizes J_μ over $L^1(F)$. Also, if there exists a unique $f^* \in BV(F)$ such that $\Phi f^* = g$, we can choose $\{\mu_N\} \subset \mathbb{R}^+$ to converge to 0 sufficiently slowly so that $\|f^* - f_N^*\|_{L^1(F)} \to 0$ as $N \to \infty$.

5 Concluding Remarks

We have studied in this paper some approximation issues in estimating images via total variation regularization. These results are applicable in a wide variety of situations, such as inverse interferometry. The issue of convergence of minimizers computed using piecewise constant basis functions is a difficult matter, one which led us to consider the use of a slightly different definition of total variation.

One criticism of the use of the 1-norm in $\mathbb{R}^d$ is its lack of rotation invariance, which means that rotated images will have different cost functional values. There is thus a price to be paid for obtaining a consistent numerical scheme. It is unclear what the practical effect of rotation invariance (or lack thereof) really means to image reconstruction. We plan to continue the investigation of the difference of these two total variation definitions.

We also remark that our preliminary numerical findings in inverse interferometry are very promising. The penalized minimal residual ideas discussed herein provide substantial improvement over the direct inversion methods based on the inversion of the density-to-phase map. A complete study of the computational aspects and numerical simulations is underway, with results to be published elsewhere.

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