Galerkin/Runge-Kutta Discretizations for Semilinear Parabolic Equations

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Abstract. A new class of fully discrete Galerkin/Runge-Kutta methods is constructed and analyzed for semilinear parabolic initial boundary value problems. Unlike any classical counterpart, this class offers arbitrarily high order convergence without suffering from what has been called *order reduction*. In support of this claim, error estimates are proved, and computational results are presented. Furthermore, it is noted that special Runge-Kutta methods allow computations to be performed in parallel so that the final execution time can be reduced to that of a low order method.

Key words: Implicit Runge-Kutta methods, semilinear, error estimates.

1980 Mathematics Subject Classification: 65M15.

1 Introduction.

In this paper, three classes of semilinear initial boundary value problems are considered. Specifically, the goal is to construct and analyze fully discrete approximations to the unique solution $u(\mathbf{x}, t)$ of:

$$\left\{\begin{array}{rcll} \partial_t u &=& -Lu + f & \text{in} \quad \Omega \times [0, t^*] \\ u &=& 0 & \text{on} \quad \partial\Omega \times [0, t^*] \\ u(\mathbf{x}, 0) &=& u^0(\mathbf{x}) & \text{in} \quad \Omega, \end{array}\right. \tag{1.1}$$

where:

$$Lu \equiv -\sum_{i,j=1}^{N} \partial_{x_i}(\ell_{ij}(\mathbf{x})\partial_{x_j} u) + \ell_0(\mathbf{x})u,$$

and f takes one of the following forms:

$$f = f(\mathbf{x}, t, u), \tag{1.1.i}$$

$$f = \mathbf{g}(\mathbf{x}, t, u) \cdot \nabla u, \tag{1.1.ii}$$

$$f = f(\mathbf{x}, t, u, \nabla u). \tag{1.1.iii}$$

Here, Ω is a bounded domain in $\mathbf{R}^N$ with $\partial\Omega$ sufficiently smooth. Also, $\ell_{ij}(\mathbf{x})$ and $\ell_0(\mathbf{x})$ are assumed to be smooth. Further, on $\bar{\Omega}$, the matrix $\{\ell_{ij}\}_{i,j=1}^N$ is symmetric and uniformly positive definite and ℓ_0 is nonegative. The initial data u^0 is assumed to be both sufficiently smooth and compatible, and precise hypotheses on the required smoothness of the solution u are made as needed. Then, in the respective sections, it is assumed that there exist constants ρ, $c_\rho > 0$ such that for all $(\mathbf{x}, t) \in \bar{\Omega} \times [0, t^*]$, one of the following local Lipschitz properties holds. Specifically, it is assumed that in case (1.1.i):

$$|u(\mathbf{x}, t) - U| \le \rho \quad \Rightarrow \quad |f(\mathbf{x}, t, u(\mathbf{x}, t)) - f(\mathbf{x}, t, U)| \le c_\rho |u(\mathbf{x}, t) - U|, \tag{1.2}$$

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This work was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665-5225.

in case (1.1.ii), for $1 \le i \le N$:

$$|u(\mathbf{x},t) - U| \le \rho \quad \Rightarrow \quad |g_i(\mathbf{x},t,u(\mathbf{x},t)) - g_i(\mathbf{x},t,U)| \le c_\rho |u(\mathbf{x},t) - U|, \tag{1.3}$$

and in case (1.1.iii), for $1 \le j \le N$:

$$\begin{array}{ccc} |u(\mathbf{x},t) - U_0| \le \rho & & |f(\mathbf{x},t,u(\mathbf{x},t),\mathbf{U}) - f(\mathbf{x},t,U_0,\mathbf{U})| \le c_\rho |u(\mathbf{x},t) - U_0| \\ \text{and} & \Rightarrow & \text{and} \\ \max\limits_{1\le i \le N} |\partial_{x_i} u(\mathbf{x},t) - U_i| \le \rho & & |f(\mathbf{x},t,U_0,\mathbf{V}^j) - f(\mathbf{x},t,U_0,\mathbf{U})| \le c_\rho |\partial_{x_j} u(\mathbf{x},t) - U_j|, \end{array} \tag{1.4}$$

where $V_i^j \equiv U_i, i \neq j$ and $V_j^j \equiv \partial_{x_j} u(\mathbf{x},t)$. In addition, to prove certain estimates for case (1.1.iii), it is assumed that:

$$\begin{array}{ccc} |u(\mathbf{x},t) - U| \le \rho & & \\ \text{and} & \Rightarrow & \max\limits_{1 \le i,j \le N} |\partial^2_{d_i d_j} f(\mathbf{x},t,U_0,\mathbf{U})| \le c_\rho \\ \max\limits_{1\le i \le N} |\partial_{x_i} u(\mathbf{x},t) - U_i| \le \rho & & \end{array} \tag{1.5}$$

where $\partial_{d_j} f$ denotes partial differentiation of f with respect to its argument connected with the jth spatial derivative of u. See the remarks prior to Proposition 4.3 for an explanation of the division (1.1.i) - (1.1.iii).

Now, for $1 \le p \le \infty$ and integers $s \ge 0$, let $W^{s,p} \equiv W^{s,p}(\Omega)$ represent the well-known Sobolev spaces consisting of functions with (distributional) derivatives of order $\le s$ in $L_p \equiv L_p(\Omega)$. Also, let $\|\cdot\|_{W^{s,p}}$ denote the usual norm. Then, in particular, take $H^s \equiv W^{s,2}$ and denote its norm by $\|\cdot\|_s$. In addition, let H_0^1 be the subspace of H^1 consisting of functions vanishing on $\partial\Omega$ in the sense of trace. Its dual is denoted by H^{-1} with norm $\|\cdot\|_{-1}$. Next, let the inner product on L_2 be denoted by $(\cdot,\cdot)$, and the associated norm by $\|\cdot\|$. Further, $\|\cdot\|_{L_\infty}$ represents the norm on L_∞, and $\|\cdot\|_{s,\infty}$ the norm on $L_\infty([0,t^*],H^s)$. See Adams [1] for more details.

Equipped with the above notation, let the following be combined with (1.3) and (1.4), respectively. Specifically, assume that in case (1.1.ii):

$$\max_{1\le i\le N} \sup_{0\le t\le t^*} \|g_i(t,u(t))\|_{W^{1,\infty}} \equiv c_g < \infty, \tag{1.6}$$

and in case (1.1.iii):

$$\max_{1\le j\le N} \sup_{0\le t\le t^*} \|\partial_{d_j} f(t,u(t),\nabla u(t))\|_{W^{1,\infty}} \equiv c_f < \infty. \tag{1.7}$$

Now, let L be extended to have domain $H^2 \cap H_0^1$. Then, L is L_2-selfadjoint and for every nonegative integer s, it is bounded from $H^{s+2} \cap H_0^1$ into H^s. Furthermore, introducing the solution operator T for the elliptic problem:

$$\left\{ \begin{array}{rcll} Lv & = & w & \text{in } \Omega \\ v & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

as $Tw \equiv v$, it is well-known (Friedman [11]) that for every nonnegative integer s, T is bounded from H^s into $H^{s+2} \cap H_0^1$. Also, the solution operator is positive definite and selfadjoint on L_2; hence, T has a square root and it can be shown (Thomée [17]) that:

$$\|T^{\frac{1}{2}} v\| \le c\|v\|_{-1} \qquad \forall v \in H^{-1}. \tag{1.8}$$

Note that here and throughout this work, c (sometimes with a subscript) is used to denote a general positive constant, not necessarily the same in any two places. Moreover, if in a given (in)equality, there is a crucial element upon which c is meant to depend, such dependence is indicated explicitly.

A rough description of the results now follows. For this, let h and k denote spatial and temporal discretization parameters respectively, and suppose that U_h^n is a fully discrete approximation to $u(nk)$ obtained according to (1.34) described below. Now, in section 2, the error committed for the approximation of the solution to (1.1) in case (1.1.i), is shown to be of optimal order in L_2:

$$\max_n \|U_h^n - u^n\| = \mathcal{O}(h^r + k^\nu) \tag{1.9}$$

under the condition that $h^{-N/2}(h^r + k^\nu)$ is sufficiently small. Here, r and ν represent respectively, optimal exponents, characteristic of the Galerkin method and the Runge-Kutta method upon which the fully discrete scheme is based. Next, section 3 deals with case (1.1.ii) and the same optimal L_2 estimate is established but under the additional condition that $h^{-N/2}k^{-1/2}(h^r + k^\nu)$ is small enough. Finally, in section 4, case (1.1.iii) is studied, and it is proved that the error is of optimal order in H^1:

$$\max_n \|U_h^n - u^n\|_1 = \mathcal{O}(h^{r-1} + k^\nu) \tag{1.10}$$

provided $h^{-N/2}(h^{r-1} + k^\nu)$ is small enough. Then, a duality argument is used to obtain (1.9). For each of the cases (1.1.i) - (1.1.iii), results for the starting scheme (1.37) are stated without proof, since what is presented for the principal scheme (1.34) captures the main ideas with fewer details. Nevertheless, complete proofs are provided in [13]. Also, in the latter, linear problems with time dependent coefficients and quasilinear problems are considered. Further, preconditioned iterative methods are combined with specially constructed Galerkin/Runge-Kutta schemes and the results obtained are similar to those reported here.

It should also be mentioned that the discovery of the methods described below was fortuitous. Note that there are extrapolation options other than (1.33) and (1.36), which are apparently more natural. For example, $\mathrm{T}^l e$ could be used instead of $l!\,A^l e$ in (1.32) and (1.35), since this is suggested by the case that $f = f(\mathbf{x}, t)$. On the other hand, an iterative procedure could be used during initial time steps to approximate fully implicit stages, and this might be followed with a standard extrapolation using previously computed stages to approximate stages for the current time step. Both of these ideas are considered in a computational section. However, under rather general conditions, optimal order convergence is proved and demonstrated computationally only for (1.34) and (1.37). In fact, for the linear nonhomogenous problem, Crouzeix [7] has developed an explicit example showing that unless $\partial_t^l u$ is in the domain of L^m for certain l and m, a classical fully discrete scheme fashioned after (1.25) cannot be expected to offer optimal order convergence.

In [3], Baker, Dougalis, and Karakashian analyze Galerkin/Multistep fully discrete approximations for the solution of (1.1) in case (1.1.i). Assuming local Lipschitz properties, they prove an optimal L_2 error estimate such as (1.9). Also, for quasilinear equations which embrace case (1.1.ii), Wheeler [18] has proved optimal L_2 estimates for several discrete time Galerkin procedures up to second order in time. Further, Bramble and Sammon* have announced related results, including an optimal L_2 estimate for a Galerkin/Obrechkoff method which is fourth order in time. Finally, note that in [10], Dougalis and Karakashian analyze Galerkin/Runge-Kutta approximations for the Korteweg-De Vries equation, and they prove optimal L_2 estimates for some *modified* IRKM's which are up to fourth order in time. Hence, the spirit of their work is similar to that of the present study.

In the remainder of this section, there is a presentation of material relevant to the spatial and temporal discretizations considered here, which concludes with a precise definition of the

*Bramble, J. H., and Sammon, P. G., "Efficient Higher Order Single Step Methods for Parabolic Problems: Part II."

schemes for which the above claims are made.

Spatial Discretizations

In terms of the solution operator, (1.1) can be written as:

$$\begin{cases} \partial_t T u &= -u + Tf \\ u(0) &= u^0. \end{cases} \tag{1.11}$$

For the spatial approximation of the solution to this problem, let $\{S_h\}_{0<h<1}$ be a family of finite-dimensional subspaces of H^1. Then, suppose that a corresponding family of operators $\{T_h\}_{0<h<1}$ is given satisfying:

i. $T_h : L_2 \to S_h$ is selfadjoint, positive semidefinite on L_2, and positive definite on S_h.

ii. There is an integer $r \geq 2$ such that:

$$\|(T - T_h)v\| + h\|(T - T_h)v\|_1 \leq ch^s \|v\|_{s-2} \qquad \forall v \in H^{s-2}, \quad 2 \leq s \leq r. \tag{1.12}$$

Now, problem (1.11) has the following semidiscrete formulation. Find $u_h : [0, t^*] \to S_h$ such that:

$$\begin{cases} \partial_t T_h u_h &= -u_h + T_h f_h \\ u_h(0) &= u_h^0 \end{cases} \tag{1.13}$$

where f_h represents f depending on u_h instead of u as indicated in one of (1.1.i) - (1.1.iii), and $u_h^0 \in S_h$ is a suitable approximation to u^0.

To make the machinery more definite, consider the following *Ordinary Galerkin Method.* From (1.1), let $D(\cdot,\cdot)$ be a symmetric bilinear form defined by:

$$D(v, w) \equiv \sum_{i,j=1}^{N} (\ell_{ij}\partial_{x_i} v, \partial_{x_j} w) + (\ell_0 v, w) \qquad v, w \in H_0^1.$$

Then, take S_h to consist of continuous, piecewise polynomials of degree $\leq r - 1$, vanishing on $\partial\Omega$. Now, let $T_h : L_2 \to S_h$ be defined by:

$$D(T_h w, \chi) \equiv (w, \chi) \qquad \forall w \in L_2, \quad \forall \chi \in S_h.$$

For more examples of Galerkin methods satisfying the assumptions above as well as others below, see Bramble, Schatz, Thomée, and Wahlbin [4], and the references cited therein.

Next, the following *inverse properties* are prescribed for S_h. Throughout this work, it is assumed that $S_h \subset L_\infty$ and:

$$\|\chi\|_{L_\infty} \leq ch^{-N/2}\|\chi\| \qquad \forall \chi \in S_h. \tag{1.14}$$

Moreover, for cases (1.1.ii) and (1.1.iii), it is assumed that $S_h \subset W^{1,\infty}$ and hence:

$$\|\chi\|_{W^{1,\infty}} \leq ch^{-N/2}\|\chi\|_1 \qquad \forall \chi \in S_h. \tag{1.15}$$

In fact, for certain estimates related to case (1.1.iii), it is assumed that:

$$\|\chi\|_{W^{1,p}} \leq ch^{-N(\frac{1}{2}-\frac{1}{p})}\|\chi\|_1 \qquad \forall \chi \in S_h, \quad p \geq 2. \tag{1.16}$$

For details connected with (1.14) - (1.16), see Ciarlet [6].

According to the properties prescribed above, the restriction of T_h to S_h is invertible and its inverse is henceforth denoted by L_h. Since L_h is also positive definite and selfadjoint on S_h, both L_h and T_h have square roots but it is also assumed that:

$$\|T_h^{\frac{1}{2}} w\|_1 \le c\|w\| \qquad \forall w \in L_2, \tag{1.17}$$

and:

$$\|\chi\|_1 \le c\|L_h^{\frac{1}{2}}\chi\| \qquad \forall \chi \in S_h. \tag{1.18}$$

Defining the *elliptic projection* operator as $P_E \equiv T_h L$, it follows from (1.12) that:

$$\|(I - P_E)v\| + h\|(I - P_E)v\|_1 \le ch^s\|v\|_s \qquad \forall v \in H^s \cap H_0^1, \quad 2 \le s \le r. \tag{1.19}$$

Also, it can be shown that $P_0 \equiv L_h T_h$ is the orthogonal projection of L_2 onto S_h. Then, since $I - P_0$ is majorized by $I - P_E$ in L_2, it follows from (1.19) that:

$$\|(I - P_0)v\| \le ch^s\|v\|_s \qquad \forall v \in H^s \cap H_0^1, \quad 2 \le s \le r. \tag{1.20}$$

In addition to (1.19), with $\omega(t) \equiv P_E u(t)$, it is assumed for case (1.1.i) that:

$$\sup_{0 \le t \le t^*} \|u(t) - \omega(t)\|_{L_\infty} \equiv \gamma_0(h) \to 0, \qquad \text{as } h \to 0. \tag{1.21}$$

Further, for cases (1.1.ii) and (1.1.iii), it is assumed that:

$$\sup_{0 \le t \le t^*} \|u(t) - \omega(t)\|_{W^{1,\infty}} \equiv \gamma_1(h) \to 0, \qquad \text{as } h \to 0. \tag{1.22}$$

For details connected with (1.21) and (1.22), see Rannacher and Scott [15], and Schatz and Wahlbin [16].

Now, problem (1.13) takes the following form. Find $u_h : [0, t^*] \to S_h$ satisfying:

$$\left\{ \begin{array}{rcl} \partial_t u_h & = & -L_h u_h + P_0 f_h \\ u_h(0) & = & u_h^0. \end{array} \right. \tag{1.23}$$

In [3], Baker, Dougalis and Karakashian analyze approximations of the form (1.23) with f as in (1.1.i). Assuming local Lipschitz properties, they prove an optimal L_2 estimate:

$$\|u - u_h\|_{0,\infty} = \mathcal{O}(h^r).$$

Also, in [18], Wheeler proves such an optimal L_2 estimate for Galerkin semidiscrete approximations for quasilinear problems which include case (1.1.ii). Then, in [9], Dendy studies such approximations for quasilinear problems which embrace case (1.1.iii). Assuming global Lipschitz properties, he establishes the following:

$$\|u - u_h\|_{1,\infty} = \mathcal{O}(h^{r-1}) \qquad \text{and} \qquad \|u - u_h\|_{0,\infty} = \mathcal{O}(h^{\min(r, 2r-2-N/2)}).$$

In the present paper, semidiscrete approximations are not analyzed. Instead, (1.23) serves only as a source of inspiration for fully discrete approximations, and u_h is not even mentioned in forthcoming proofs.

Temporal Discretizations

For the temporal approximation of the solution to (1.23), *Implicit Runge-Kutta Methods* (IRKM's) are now introduced. Given an integer $q \ge 1$, a *q-stage* IRKM is characterized by a set of constants:

$$
\begin{array}{ccc|c}
a_{11} & \cdots & a_{1q} & \tau_1 \\
\vdots & & \vdots & \vdots \\
a_{q1} & \cdots & a_{qq} & \tau_q \\
\hline
b_1 & \cdots & b_q &
\end{array}
$$

and it is convenient to make the following definitions:

$$A \equiv \{a_{ij}\}_{1\le i,j\le q}, \quad \mathrm{T} \equiv \operatorname*{diag}_{1\le i\le q}\{\tau_i\}, \quad b^T \equiv \langle b_1, b_2, \ldots, b_q\rangle, \quad e^T \equiv \langle 1, 1, \ldots, 1\rangle.$$

For the IRKM formulation used in this work, choose arbitrarily, $t_0 \in \mathbf{R}$, $\mathbf{y}_0 \in \mathbf{R}^n$, $\mathbf{F} : \mathbf{R}^{n+1} \to \mathbf{R}^n$ sufficiently smooth, and $k > 0$ sufficiently small, so that for $t_0 \le t \le t_0+k$, smooth functions $\mathbf{y}, \hat{\mathbf{y}} : \mathbf{R} \to \mathbf{R}^n$ are well-defined by:

$$\begin{cases} D_t\mathbf{y}(t) &= \mathbf{F}(t, \mathbf{y}(t)) \\ \mathbf{y}(t_0) &= \mathbf{y}_0, \end{cases} \tag{1.24}$$

and:

$$\begin{cases} \mathbf{y}^i(t) &= \mathbf{y}_0 + (t-t_0)\sum_{j=1}^{q} a_{ij}\mathbf{F}(t_0 + \tau_j(t-t_0), \mathbf{y}^j(t)), \quad 1 \le j \le q \\ \hat{\mathbf{y}}(t) &= \mathbf{y}_0 + (t-t_0)\sum_{i=1}^{q} b_i\mathbf{F}(t_0 + \tau_i(t-t_0), \mathbf{y}^i(t)). \end{cases} \tag{1.25}$$

The method is described as *explicit* if $a_{ij} = 0$, $i \le j$ and *implicit* if for any i, $a_{ii} \ne 0$. Also, it is said to have *order* ν if for every $\mathbf{y}$ and $\hat{\mathbf{y}}$ defined as above, $D_t^l\mathbf{y}(t_0) = D_t^l\hat{\mathbf{y}}(t_0)$, $0 \le l \le \nu$. Butcher [5] has developed simple conditions for the above parameters, which guarantee a given order; however, only the following is explicitly required in this work:

$$l!\, b^T A^{l-1} e = 1 \qquad 1 \le l \le \nu. \tag{1.26}$$

To see the roots of condition (1.26), let (1.24) have $n = 1$, $t_0 = 0$, $y_0 = 1$, and $F(y) = -y$, so that $y(t) = e^{-t}$. Then, from (1.25), $\hat{y}(t) = r(t)$ where $r(z)$ is a rational approximation to the exponential e^{-z} given by:

$$r(z) \equiv 1 - z b^T(I + zA)^{-1}e. \tag{1.27}$$

Expanding this expression shows that $r(z)$ is a νth order approximation to the exponential if and only if (1.26) holds. Next, with regard to stability, an IRKM is said to be A_0-*stable* if:

$$|r(z)| \le 1 \qquad \forall z \ge 0, \tag{1.28}$$

and *strongly* A_0-*stable* if:

$$\sup_{z\ge z_0} |r(z)| < 1 \qquad \forall z_0 > 0. \tag{1.29}$$

The former is required of all IRKM's considered here, but for cases (1.1.ii) and (1.1.iii), the latter is assumed. Note that the spectrum of A, $\sigma(A)$ is related to the poles of $r(z)$ and in addition to the above, it is assumed throughout this work that:

$$\sigma(A) \subset \{z \in \mathbf{C} : \Re z \ge 0, z \ne 0\}. \tag{1.30}$$

Finally, for cases (1.1.ii) and (1.1.iii), optimal results seem to require the mild condition that:

$$h^2 \le ck \tag{1.31}$$

unless $S_h \subset H_0^1$. The approach used is preferred since other attempts have led to (1.31) regardless of the boundary behavior of functions in S_h.

Returning to the temporal discretization of (1.23), let a q-stage IRKM of order $\nu \geq 1$ be given. Then, the constants $\{\alpha_{jm}\}_{1\leq j\leq q}^{0\leq m\leq \nu-1}$ are well-defined by:

$$\sum_{m=0}^{\nu-1} \alpha_{jm} m^l = (-1)^l l! \hat{e}_j^T A^l e \qquad 0 \leq l \leq \nu-1, \quad 1 \leq j \leq q \quad (m^l|_{m=l=0} \equiv 1) \tag{1.32}$$

since their computation involves the inversion of the $\nu \times \nu$ Vandermonde matrix $\{m^l\}_{m,l=0}^{\nu-1}$. Next, with $n^* k \equiv t^*$, and $t^n \equiv nk$, for $\nu - 1 \leq n \leq n^* - 1$, suppose the approximations $\{U_h^m\}_{m=0}^n \subset S_h$ are given, where $U_h^m \simeq u^m$ and $u^m \equiv u(\mathbf{x}, t^m)$. Now, with $\mathbf{L}_2 \equiv [L_2]^q$, define the extrapolation operators $\mathcal{E}^n : [L_2]^\nu \to \mathbf{L}_2$ to have components:

$$\mathcal{E}_j^n f_h \equiv \sum_{m=0}^{\nu-1} \alpha_{jm} f(\mathbf{x}, t^{n-m}, U_h^{n-m}, \nabla U_h^{n-m}) \tag{1.33}$$

with appropriate modifications for cases (1.1.i) and (1.1.ii). Also, with $\mathbf{S}_h \equiv [S_h]^q$, define $\mathcal{L}_h : \mathbf{S}_h \to \mathbf{S}_h$ and $\mathcal{P}_0 : \mathbf{L}_2 \to \mathbf{S}_h$ by:

$$\mathcal{L}_h \equiv \operatorname*{diag}_{q\times q}\{L_h\} \qquad \text{and} \qquad \mathcal{P}_0 \equiv \operatorname*{diag}_{q\times q}\{P_0\}.$$

Finally, let $U_h^{n+1} \simeq u^{n+1}$ be given by what is henceforth called the **principal scheme**:

$$\begin{cases} \bar{U}_h^n &= eU_h^n - kA\mathcal{L}_h \bar{U}_h^n + kA\mathcal{P}_0 \mathcal{E}^n f_h \\ U_h^{n+1} &= (I - b^T A^{-1} e) U_h^n + b^T A^{-1} \bar{U}_h^n \end{cases} \tag{1.34}$$

where $\bar{U}_h^n \in \mathbf{S}_h$ and $\mathcal{E}^n f_h$ respectively, are well-defined provided $[I + kA\mathcal{L}_h]$ is invertible and the approximations are sufficiently accurate. Here, $A\mathcal{L}_h$ for example, is understood in the sense of composition of operators defined on $\mathbf{S}_h$. Note that (1.23) and (1.34) are only partially patterned after (1.24) and (1.25), i. e., the stages $\bar{U}_h^n$ are not fully implicit, and extrapolation circumvents the solution of a nonlinear system of algebraic equations for each n.

Since the extrapolation for the principal scheme uses previously computed approximations, a starting procedure is required to generate $\{U_h^m\}_{m=0}^{\nu-1}$. Hence, for $i \geq 1$, define $\nu_i \equiv \min(\nu, i)$, and as with (1.32), let the constants $\{\alpha_{jm}^{n,i}\}_{0\leq m\leq \nu_i-1, 1\leq j\leq q}^{0\leq n\leq \min(i-1,\nu-2)}$ be determined by:

$$\sum_{m=0}^{\nu_i-1} \alpha_{jm}^{n,i} (m-n)^l = l! \hat{e}_j^T A^l e \qquad 0 \leq l \leq \nu_i - 1, \;\; 1 \leq j \leq q \quad ((m-n)^l|_{m-n=l=0} \equiv 1). \tag{1.35}$$

Then, with $0 \leq n \leq \min(i-1, \nu-2)$, and $m_j \equiv \min(j, \nu-1), 1 \leq j < i$, $m_j \equiv n, j = i$, suppose the approximations $\{U_h^{m,j}\}_{0\leq m\leq m_j}^{1\leq j\leq i} \subset S_h$ are given, where $U_h^{m,j} \simeq u^m$. Next, define the extrapolation operators $\mathcal{E}^{n,i} : [L_2]^{\nu_i} \to \mathbf{L}_2$ to have components:

$$\mathcal{E}_j^{n,i} f_h \equiv \begin{cases} \displaystyle\sum_{m=0}^{n} \alpha_{jm}^{n,i} f(\mathbf{x}, t^m, U_h^{m,i}, \nabla U_h^{m,i}) + \sum_{m=n+1}^{\nu_i-1} \alpha_{jm}^{n,i} f(\mathbf{x}, t^m, U_h^{m,i-1}, \nabla U_h^{m,i-1}), \\ \hfill 0 \leq n \leq \nu_i - 2 \\ \displaystyle\sum_{m=0}^{\nu_i-1} \alpha_{jm}^{n,i} f(\mathbf{x}, t^m, U_h^{m,i}, \nabla U_h^{m,i}), \qquad n = i-1 \leq \nu-2 \end{cases} \tag{1.36}$$

with appropriate modifications for cases (1.1.i) and (1.1.ii). Finally, let $U_h^{n+1,i} \simeq u^{n+1}$ be given by what is henceforth called the **starting scheme**:

$$\left\{\begin{array}{lcl} \bar{U}_h^{n,i} & = & eU_h^{n,i} - kA\mathcal{L}_h\bar{U}_h^{n,i} + kA\mathcal{P}_0\mathcal{E}^{n,i}f_h \\ U_h^{n+1,i} & = & (I - b^TA^{-1}e)U_h^{n,i} + b^TA^{-1}\bar{U}_h^{n,i}. \end{array}\right. \tag{1.37}$$

With regard to the initial data $U_h^{0,j} \equiv U_h^0$, $j \geq 0$, for case (1.1.i), the following is sufficient:

$$U_h^0 = P_0 u^0. \tag{1.38}$$

However, for cases (1.1.ii) and (1.1.iii), it is required that:

$$U_h^0 = [I + kL_h]^{-1}P_0[I + kL]u^0. \tag{1.39}$$

The cases (1.1.i) - (1.1.iii) are now analyzed separately in the next three sections.

2 Semilinearities Independent of Spatial Derivatives.

In this section, the principal scheme (1.34) is analyzed for the approximation of the solution to (1.1) in case (1.1.i), and (1.9) is established. That the stages are well-defined depends on the Lemma below. Its proof involves a spectral argument after A is transformed to Jordan form, and the details are provided by Karakashian [12]. First, let the following be defined in the natural way for the product spaces:

$$(\mathbf{\Phi}, \mathbf{\Psi}) \equiv \sum_{i=1}^{q}(\phi_i, \psi_i), \qquad \|\Phi\| \equiv (\Phi, \Phi)^{\frac{1}{2}} \qquad \Phi, \Psi \in \mathbf{L}_2$$

Lemma 2.1 *Provided (1.30) holds, $[I + kA\mathcal{L}_h]$ is invertible and:*

$$\|(k\mathcal{L}_h)^\theta[I + kA\mathcal{L}_h]^{-1}X\| \leq c\|X\| \qquad \forall X \in \mathbf{S}_h, \ \ 0 \leq \theta \leq 1. \tag{2.1}$$

■

Now, sufficiently accurate starting approximations are assumed given and for $\nu - 1 \leq n \leq n^* - 1$, an error equation relating $U_h^n - \omega^n$ to $U_h^{n+1} - \omega^{n+1}$ appears below. For the sequel, make the definitions:

$$\xi^n \equiv U_h^n - \omega^n, \qquad \eta^n \equiv u^n - \omega^n, \qquad f^n \equiv f(\mathbf{x}, t^n, u^n),$$

$$\bar{u}^n \equiv \sum_{l=0}^{\nu} A^l e \partial_t^l u^n k^l, \qquad \mathcal{E}_j^n f \equiv \sum_{l=0}^{\nu-1}\alpha_{jm}f^{n-m}, \qquad f_h^n \equiv f(\mathbf{x}, t^n, U_h^n),$$

$$\mathcal{L} \equiv \operatorname*{diag}_{q\times q}\{L\}, \qquad \mathcal{P}_E \equiv \operatorname*{diag}_{q\times q}\{P_E\}, \qquad r_h \equiv I - kb^T\mathcal{L}_h[I + kA\mathcal{L}_h]^{-1}e.$$

After some straightforward calculations, the following error equation is obtained:

$$\begin{aligned} \xi^{n+1} = r_h\xi^n \quad & + \quad kb^T\mathcal{L}_h[I + kA\mathcal{L}_h]^{-1}\mathcal{P}_0\{\bar{u}^n - eu^n + kA\mathcal{L}\bar{u}^n - kA\mathcal{E}^n f\} \\ & - \quad kb^T\mathcal{L}_h[I + kA\mathcal{L}_h]^{-1}(\mathcal{P}_0 - \mathcal{P}_E)(\bar{u}^n - eu^n) \\ & + \quad (P_0 - P_E)(u^{n+1} - u^n) \\ & - \quad P_0\{u^{n+1} - u^n + kb^T\mathcal{L}\bar{u}^n - kb^T\mathcal{E}^n f\} \\ & + \quad kb^T[I + kA\mathcal{L}_h]^{-1}\mathcal{P}_0[\mathcal{E}^n f_h - \mathcal{E}^n f] \quad \equiv \quad r_h\xi^n + \sum_{l=1}^{4}\psi_l^n + \phi^n, \\ & \qquad \nu - 1 \leq n \leq n^* - 1. \end{aligned} \tag{2.2}$$

Stability is established as follows.

Proposition 2.1 *If the rational function (1.27) satisfies (1.28), then there exists a constant $\tilde{c} \leq 0$, such that:*

$$\|r_h \chi\| \leq (1 + \tilde{c}k)\|\chi\| \qquad \forall \chi \in S_h. \tag{2.3}$$

Furthermore, $\tilde{c} < 0$ if (1.29) is satisfied, and k is small enough.

Proof: Since $\nu \geq 1$, $r(0) = 1 = -r'(0)$. So, let $z_1, \theta > 0$ be chosen so that:

$$|r(z)| \leq 1 - \theta z \qquad 0 \leq z \leq z_1.$$

By (1.18), $\sigma(kL_h) \subset [ck, \infty)$. So using a spectral argument, it follows that for k small enough, there is a $\tilde{c} \leq 0$ such that (2.3) is satisfied and $\tilde{c} < 0$ if (1.29) holds. ∎

Next, the order of consistency is established in the following. Also, ϕ^n is majorized by terms which are summed out in the convergence proof.

Proposition 2.2 *The terms $\{\psi_l^n\}_{l=1}^4$ of (2.2) satisfy:*

$$\sum_{l=1}^{4} \|\psi_l^n\| \leq ck(h^r + k^\nu)\{\|\partial_t^{\nu+1} u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty} + \|\partial_t u\|_{r,\infty}\}. \tag{2.4}$$

Also, if $\{U_h^{n-m}\}_{m=0}^{\nu-1}$ are given satisfying:

$$\max_{0 \leq m \leq \nu-1} \|U_h^{n-m} - u^{n-m}\|_{L_\infty} \leq \rho, \tag{2.5}$$

then ϕ^n of (2.2) satisfies:

$$\|\phi^n\| \leq ckh^r \|u\|_{r,\infty} + ck \sum_{m=0}^{\nu-1} \|\xi^{n-m}\|. \tag{2.6}$$

Proof: The terms of (2.2) are considered in the order in which they appear. Since for $0 \leq l \leq \nu$, $\partial_t^l u \in H^2 \cap H_0^1$, by (1.1):

$$\begin{aligned} \bar{u}^n - eu^n + kA\mathcal{L}\bar{u}^n &= \sum_{l=1}^{\nu} A^l e \partial_t^l u^n k^l + \sum_{l=0}^{\nu} A^{l+1} e[L\partial_t^l u^n] k^{l+1} \\ &= -A^{\nu+1} e \partial_t^{\nu+1} u^n k^{\nu+1} + kA \sum_{l=0}^{\nu} A^l e \partial_t^l f^n k^l. \end{aligned}$$

By (1.32):

$$\mathcal{E}^n f = \sum_{l=0}^{\nu-1} A^l e \partial_t^l f^n k^l + E \tag{2.7}$$

where E has components:

$$E_j \equiv \frac{1}{(\nu-1)!} \sum_{m=0}^{\nu-1} \alpha_{jm} \int_{t^n}^{t^{n-m}} (t^{n-m} - s)^{\nu-1} \partial_s^\nu f(s, u(s)) ds.$$

Now with (2.1) and (1.1), it follows that:

$$\|\psi_1^n\| \leq ck^{\nu+1}\{\|\partial_t^{\nu+1} u\|_{0,\infty} + \|\partial_t^\nu f\|_{0,\infty}\} \leq ck^{\nu+1}\{\|\partial_t^{\nu+1} u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty}\}. \tag{2.8}$$

Next, as with (1.32), the constants $\{\beta_{jm}\}_{1\leq j\leq q}^{0\leq m\leq \nu}$ are well-defined by:

$$\sum_{m=0}^{\nu} \beta_{jm} m^l = l!\nu^l \hat{e}_j^T A^l e(1-\delta_{l0}) \qquad 0\leq l\leq \nu, \quad 1\leq j\leq q \quad (m^l|_{m=l=0}\equiv 1)$$

where δ_{ij} is the Kronecker delta. Hence, define the extrapolation operator $\mathcal{X}^n$ to have components:

$$\mathcal{X}_j^n u \equiv \sum_{m=0}^{\nu} \beta_{jm} u(t^n + \frac{mk}{\nu}) \qquad 1\leq j\leq q$$

so that for $0\leq p\leq \nu$, with ill-defined sums understood to be zero:

$$\mathcal{X}^n u = \sum_{l=1}^{p} A^l e \partial_t^l u^n k^l + F^p$$

where F^p has components:

$$F_j^p = \frac{1}{p!}\sum_{m=0}^{\nu} \beta_{jm} \int_{t^n}^{t^n+\frac{mk}{\nu}} (t^n + \frac{mk}{\nu} - s)^p \partial_s^{p+1} u(s) ds \qquad 1\leq j\leq q.$$

Now note that ψ_2^n is given by:

$$\begin{aligned}\psi_2^n &= kb^T[I+kA\mathcal{L}_h]^{-1}\mathcal{P}_0\mathcal{L}[\bar{u}^n - eu^n - \mathcal{X}^n u] \\ &\quad -kb^T\mathcal{L}_h[I+kA\mathcal{L}_h]^{-1}\{\mathcal{P}_0[\bar{u}^n - eu^n - \mathcal{X}^n u] + [\mathcal{P}_0 - \mathcal{P}_E]\mathcal{X}^n u\}.\end{aligned}$$

With (2.1), (1.19), and (1.20), it follows that:

$$\begin{aligned}\|\psi_2^n\| &\leq ck\|\mathcal{L}[A^\nu e\partial_t^\nu u^n k^\nu - F^{\nu-1}]\| + c\|F^\nu\| + c\|[\mathcal{P}_0 - \mathcal{P}_E]F^0\| \\ &\leq ck(k^\nu + h^r)\{\|\partial_t^\nu u\|_{2,\infty} + \|\partial_t^{\nu+1}u\|_{0,\infty} + \|\partial_t u\|_{r,\infty}\}.\end{aligned} \tag{2.9}$$

Now since:

$$\psi_3^n = \int_{t^n}^{t^{n+1}} (P_0 - P_E)\partial_s u(s) ds,$$

with (1.19) and (1.20), it follows that:

$$\|\psi_3^n\| \leq ckh^r\|\partial_t u\|_{r,\infty}. \tag{2.10}$$

Next, using (1.1), (2.7), and (1.26), the following is obtained:

$$\begin{aligned}\psi_4^n &= -P_0\{G + \sum_{l=0}^{\nu-1}\partial_t^{l+1}u^n \frac{k^{l+1}}{(l+1)!} + \sum_{l=0}^{\nu} b^T A^l e[L\partial_t^l u^n]k^{l+1} - \sum_{l=0}^{\nu-1} b^T A^l e\partial_t^l f^n k^{l+1} - kb^T E\} \\ &= -P_0\{G + b^T A^\nu e L\partial_t^\nu u^n k^{\nu+1} - kb^T E\}\end{aligned}$$

where:

$$G \equiv \frac{1}{\nu!}\int_{t^n}^{t^{n+1}} (t^{n+1}-s)^\nu \partial_s^{\nu+1} u(s) ds.$$

Then using (1.1):

$$\begin{aligned}\|\psi_4^n\| &\leq ck^{\nu+1}\{\|\partial_t^{\nu+1}u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty} + \|\partial_t^\nu f\|_{0,\infty}\} \\ &\leq ck^{\nu+1}\{\|\partial_t^{\nu+1}u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty}\}.\end{aligned} \tag{2.11}$$

Now, (2.4) follows after combining (2.8) - (2.11). Finally, because of (2.5), (1.2), and (2.1):

$$\|\phi^n\| \leq ck \sum_{m=0}^{\nu-1} \|f_h^{n-m} - f^{n-m}\| \leq cc_\rho k \sum_{m=0}^{\nu-1} \|\xi^{n-m} - \eta^{n-m}\|$$

and (2.6) follows with (1.19). ■

Now set:

$$\mathcal{N}(u) \equiv \|u\|_{r,\infty} + \|\partial_t u\|_{r,\infty} + \sum_{l=1}^{\nu} \|\partial_t^l u\|_{2,\infty} + \|\partial_t^{\nu+1} u\|_{0,\infty}$$

and (1.9) is finally established in the following.

Theorem 2.1 *Assume that (1.28) and (1.30) hold. Suppose $\{U_h^m\}_{m=0}^{\nu-1}$ are given satisfying:*

$$\max_{0\leq m\leq \nu-1} \|U_h^m - \omega^m\| \leq c(h^r + k^\nu)\mathcal{N}(u). \tag{2.12}$$

Then, provided $h^{-N/2}(h^r + k^\nu) + \gamma_0(h)$ is sufficiently small, $\{U_h^n\}_{n=\nu}^{n^*}$ are well-defined by (1.34) and the following holds:

$$\max_{0\leq n\leq n^*} \|U_h^n - u^n\| \leq c(h^r + k^\nu)\mathcal{N}(u). \tag{2.13}$$

Proof: Set $\theta(h,k) \equiv h^{-N/2}(h^r + k^\nu) + \gamma_0(h)$. It is first established that for $\theta(h,k)$ small enough:

$$\max_{0\leq m\leq n^*} \|U_h^m - u^m\|_{L_\infty} \leq \rho. \tag{2.14}$$

Note that by (1.14), (2.12), and (1.21), for $\theta(h,k)$ small enough:

$$\|U_h^m - u^m\|_{L_\infty} \leq ch^{-N/2}\|\xi^m\| + \|\eta^m\|_{L_\infty} \leq c\theta(h,k) \leq \rho \qquad 0 \leq m \leq \nu - 1.$$

Now suppose that for each h and k, there exists an $n = n(h,k)$ such that $\nu - 1 \leq n \leq n^* - 1$ and:

$$\max_{0\leq l\leq n} \|U_h^l - u^l\|_{L_\infty} \leq \rho \tag{2.15}$$

while:

$$\|U_h^{n+1} - u^{n+1}\|_{L_\infty} > \rho. \tag{2.16}$$

Given (2.15), inequalities (2.3), (2.4), and (2.6) can be combined for the error equation (2.2) to obtain:

$$\|\xi^{l+1}\| \leq (1 + \tilde{c}k)\|\xi^l\| + c_1 k(h^r + k^\nu)\mathcal{N}(u) + c_2 k \sum_{m=0}^{\nu-1} \|\xi^{l-m}\| \qquad \nu - 1 \leq l \leq n.$$

After summing this over $\nu - 1 \leq l \leq n$ and applying the discrete Gronwall Lemma to the result, it follows that:

$$\|\xi^{n+1}\| \leq (1 + c_3 k)^{n-\nu+1}\{\|\xi^{\nu-1}\| + c_1 t^*(h^r + k^\nu)\mathcal{N}(u) + c_3 k \sum_{m=0}^{\nu-1} \|\xi^m\|\} \tag{2.17}$$

for $c_3 \equiv \tilde{c} + c_2\nu \geq 0$. Also, the exponential dependence on t^* can be eliminated if $c_3 < 0$ [13]. Now by (1.14), (2.17), (2.12), and (1.21), for $\theta(h,k)$ small enough:

$$\|U_h^{n+1} - u^{n+1}\|_{L_\infty} \leq ch^{-N/2}\|\xi^{n+1}\| + \|\eta^{n+1}\|_{L_\infty} \leq c\theta(h,k) \leq \rho.$$

This contradicts (2.16), and hence, (2.14) is established. In fact, (2.13) follows from (2.12) and (2.17) after using (1.19). ■

See [13] for a proof of the following.

Theorem 2.2 *Assume that (1.28) and (1.30) hold. Then provided $h^{-N/2}(h^r + k^2) + \gamma_0(h)$ is sufficiently small, $\{U_h^{n,i}\}_{0\le n\le i}^{1\le i\le \nu-1}$ are well-defined by (1.37) and (1.38), and the following holds:*

$$\max_{0\le n\le i} \|U_h^{n,i} - \omega^n\| \le c(h^r + k^{i+1})\mathcal{N}(u) \qquad 1 \le i \le \nu - 1.$$

Therefore, (2.12) follows with $U_h^n \equiv U_h^{n,\nu-1}$, $0 \le n \le \nu - 1$. ∎

3 Semilinearities Depending Linearly on Spatial Derivatives.

In this section, the principal scheme (1.34) is analyzed for the approximation of the solution to (1.1) in case (1.1.ii), and (1.9) is established. Again, sufficiently accurate starting approximations are assumed given. Also, an error equation appears below, which differs in certain ways from (2.2). For the sequel, define the new terms:

$$\bar{u}^n \equiv \sum_{l=0}^{\nu-1} A^l e \partial_t^l u^n k^l, \qquad \mathbf{g}^n \equiv \mathbf{g}(\mathbf{x}, t^n, u^n), \qquad f^n \equiv \mathbf{g}^n \cdot \nabla u^n, \qquad \mathbf{g}_h^n \equiv \mathbf{g}(\mathbf{x}, t^n, U_h^n).$$

Note that here, $\bar{u}^n$ contains one less term than its counterpart in (2.2). After some straightforward calculations, the following error equation is obtained:

$$\begin{aligned}
\xi^{n+1} = r_h \xi^n \quad &+ \quad k b^T [I + kA\mathcal{L}_h]^{-1} \mathcal{P}_0 \mathcal{L}\{\bar{u}^n - e u^n - kA \sum_{l=0}^{\nu-1} A^l e \partial_t^{l+1} u^n k^l\} \\
&+ \quad k b^T [I + kA\mathcal{L}_h]^{-1} \mathcal{P}_0 \{\mathcal{E}^n f - \sum_{l=0}^{\nu-1} A^l e \partial_t^l f^n k^l\} \\
&+ \quad k b^T [I + kA\mathcal{L}_h]^{-1} (\mathcal{P}_0 - \mathcal{P}_E) \sum_{l=0}^{\nu-1} A^l e \partial_t^{l+1} u^n k^l \\
&- \quad \{\omega^{n+1} - \omega^n - k b^T \sum_{l=0}^{\nu-1} A^l e \partial_t^{l+1} \omega^n k^l\} \\
&+ \quad k b^T [I + kA\mathcal{L}_h]^{-1} \mathcal{P}_0 [\mathcal{E}^n f_h - \mathcal{E}^n f] \quad \equiv \quad r_h \xi^n + \sum_{l=1}^{4} \psi_l^n + \phi^n, \\
&\qquad\qquad \nu - 1 \le n \le n^* - 1.
\end{aligned} \tag{3.1}$$

Now for the present section, stability is established in the norm:

$$|||\chi|||_1 \equiv \{(\chi, \chi) + k(L_h \chi, \chi)\}^{\frac{1}{2}} \qquad \chi \in S_h.$$

Proposition 3.1 *Suppose that (1.29) is satisfied. Then for k small enough, there exists a constant $\tilde{c} < 0$ such that the following holds for (3.1):*

$$|||\xi^{n+1}|||_1^2 \le (1 + \tilde{c}k)|||\xi^n|||_1^2 + ck^{-1} \sum_{l=1}^{4} |||\psi_l^n|||_1^2 + c\{\|(kL_h)^{-\frac{1}{2}} \phi^n\|^2 + \|(kL_h)^{\frac{1}{2}} \phi^n\|^2\}. \tag{3.2}$$

Proof: Let z_1, $\theta > 0$ be chosen as in the proof of Proposition 2.1. Next, define:

$$r_\epsilon(z) \equiv \frac{1+(1+\epsilon)z}{1+z}$$

and fix $\epsilon > 0$ small enough so that with (1.29):

$$|r_\epsilon(z) r(z)| \leq (1+\epsilon z)(1-\theta z) < 1 \qquad 0 < z \leq z_1,$$

and:

$$|r_\epsilon(z) r(z)| \leq (1+\epsilon) \sup_{z \geq z_1} |r(z)| < 1 \qquad z_1 \leq z.$$

Then just as with (2.3), there is a constant $\tilde{c} < 0$ such that:

$$\|r_\epsilon(kL_h) r_h \chi\|^2 \leq (1+\tilde{c}k)\|\chi\|^2 \qquad \forall \chi \in S_h. \tag{3.3}$$

Now set $c_\epsilon \equiv 1+\epsilon$, operate on both sides of (3.1) with $[I + c_\epsilon k L_h]$ and integrate against ξ^{n+1}. Estimation of the resulting terms follows.

With $\chi^n \equiv [I + kL_h]^{\frac{1}{2}} \xi^n$, it follows from (3.3) that:

$$([I + c_\epsilon k L_h] r_h \xi^n, \xi^{n+1}) = (r_\epsilon(kL_h) r_h \chi^n, \chi^{n+1}) \leq \tfrac{1}{2}(1+\tilde{c}k)|||\xi^n|||_1^2 + \tfrac{1}{2}|||\xi^{n+1}|||_1^2.$$

By (1.18):

$$\begin{aligned} ([I + c_\epsilon k L_h]\psi_l^n, \xi^{n+1}) &= (\{k^{-\frac{1}{2}} I + c_\epsilon (kL_h)^{\frac{1}{2}}\}\psi_l^n, k^{\frac{1}{2}}\{I + L_h^{\frac{1}{2}}\}\xi^{n+1}) \\ &\leq ck^{-1}|||\psi_l^n|||_1^2 + \tfrac{1}{16}\epsilon k \|L_h^{\frac{1}{2}} \xi^{n+1}\|^2 \qquad 1 \leq l \leq 4. \end{aligned}$$

Finally:

$$\begin{aligned} ([I + c_\epsilon k L_h]\phi^n, \xi^{n+1}) &= (\{(kL_h)^{-\frac{1}{2}} + c_\epsilon (kL_h)^{\frac{1}{2}}\}\phi^n, (kL_h)^{\frac{1}{2}}\xi^{n+1}) \\ &\leq c\{\|(kL_h)^{-\frac{1}{2}}\phi^n\|^2 + \|(kL_h)^{\frac{1}{2}}]\phi^n\|^2\} + \tfrac{1}{2}\epsilon k \|L_h^{\frac{1}{2}}\xi^{n+1}\|^2. \end{aligned}$$

Combining the above inequalities:

$$\begin{aligned} (\xi^{n+1}, \xi^{n+1}) + (1+\epsilon)k(L_h \xi^{n+1}, \xi^{n+1}) &\leq \tfrac{1}{2}(1+\tilde{c}k)|||\xi^n|||_1^2 + \tfrac{1}{2}|||\xi^{n+1}|||_1^2 + ck^{-1}\sum_{l=1}^{4}|||\psi_l^n|||_1^2 \\ &+ c\{\|(kL_h)^{-\frac{1}{2}}\phi^n\|^2 + \|(kL_h)^{\frac{1}{2}}\phi^n\|^2\} + \epsilon k \|L_h^{\frac{1}{2}}\xi^{n+1}\|^2 \end{aligned}$$

and (3.2) follows readily. ■

Now an analogue to Proposition 2.2 must be established for the stronger norm.

Proposition 3.2 *Assume that (1.31) holds. Then the terms $\{\psi_l^n\}_{l=1}^4$ of (3.1) satisfy:*

$$\sum_{l=1}^{4}|||\psi_l^n|||_1 \leq ck(h^r + k^\nu)\{\|\partial_t^{\nu+1}u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty} + \|\partial_t u\|_{r,\infty}\}. \tag{3.4}$$

Also, if $\{U_h^{n-m}\}_{m=0}^{\nu-1}$ are given satisfying:

$$\max_{0 \leq m \leq \nu-1} \|U_h^{n-m} - u^{n-m}\|_{L_\infty} \leq \rho, \tag{3.5}$$

and:

$$\max_{0\le m\le \nu-1}\|U_h^{n-m}\|_{W^{1,\infty}} \le \bar{c}, \tag{3.6}$$

for some $\bar{c} > c_u \equiv \sup_{0\le t\le t^*}\|u(t)\|_{W^{1,\infty}}$*, then* ϕ^n *of (3.1) satisfies:*

$$\|(kL_h)^{-\frac{1}{2}}\phi^n\|^2 + \|(kL_h)^{\frac{1}{2}}\phi^n\|^2 \le ck(h^r\|u\|_{r,\infty})^2 + ck\sum_{m=0}^{\nu-1}|||\xi^{n-m}|||_1^2. \tag{3.7}$$

Proof: By (2.1):

$$|||\psi_1^n|||_1 \le ck\|\mathcal{L}A^\nu e\partial_t^\nu u^n k^\nu\| \le ck^{\nu+1}\|\partial_t^\nu u\|_{2,\infty} \tag{3.8}$$

Next, by (1.32), an analogue to (2.7) follows. Hence, by (2.1) and (1.1):

$$|||\psi_2^n|||_1 \le ck^{\nu+1}\|\partial_t^\nu f\|_{0,\infty} \le ck^{\nu+1}\{\|\partial_t^{\nu+1}u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty}\}. \tag{3.9}$$

Now, recalling the development prior to (2.9), with (2.1) and (1.30), the following is obtained:

$$|||\psi_3^n|||_1 \le c\|(\mathcal{P}_E - \mathcal{P}_0)[A^\nu e\partial_t^\nu u^n k^\nu - F^{\nu-1}]\| + c\|(\mathcal{P}_E - \mathcal{P}_0)F^0\|.$$

Using $s=2$ and $s=r$ in (1.19) and (1.20), with (1.31) it follows that:

$$|||\psi_3^n|||_1 \le ch^2k^\nu\|\partial_t^\nu u\|_{2,\infty} + ckh^r\|\partial_t u\|_{r,\infty} \le ck(k^\nu + h^r)\{\|\partial_t^\nu u\|_{2,\infty} + \|\partial_t u\|_{r,\infty}\} \tag{3.10}$$

Next, set:

$$H \equiv u^{n+1} - u^n - kb^T\sum_{l=0}^{\nu-1}A^l e\partial_t^{l+1}u^n k^l$$

so that $\psi_4^n = P_E H$ and:

$$\begin{aligned} |||\psi_4^n|||_1^2 &\le \|\psi_4^n\|^2 + k\|L_h\psi_4^n\|\|\psi_4^n\| \le \tfrac{3}{2}\|P_E H\|^2 + \tfrac{1}{2}k^2\|P_0 LH\|^2 \\ &\le c\|H\|^2 + c\|(P_E - I)H\|^2 + ck^2\|H\|_2^2. \end{aligned}$$

This estimation is then continued using $s=2$ in (1.19) and applying (1.31):

$$|||\psi_4^n|||_1 \le c\|H\| + c(h^2+k)\|H\|_2 \le c\|H\| + ck\|H\|_2.$$

By (1.26):

$$H = G^p + \sum_{l=1}^{p}\partial_t^l u^n\frac{k^l}{l!} - \sum_{l=0}^{\nu-1}b^TA^le\partial_t^{l+1}u^nk^{l+1} = G^p - (\nu-p)\partial_t^\nu u^n\frac{k^\nu}{\nu!} \qquad p = \nu-1,\nu$$

where:

$$G^p \equiv \frac{1}{p!}\int_{t^n}^{t^{n+1}}(t^{n+1}-s)^p\partial_s^{p+1}u(s)ds.$$

Hence:

$$|||\psi_4^n|||_1 \le c\|G^\nu\| + ck\|G^{\nu-1} - \partial_t^\nu u^n\frac{k^\nu}{\nu!}\|_2 \le ck^{\nu+1}\{\|\partial_t^{\nu+1}u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty}\}. \tag{3.11}$$

Now, (3.4) follows after combining (3.8) - (3.11). Turning then to the Gronwall terms, by (1.17) and (2.1):

$$\begin{aligned} \|(kL_h)^\theta\phi^n\|^2 &\le ck\sum_{m=0}^{\nu-1}\|(\mathbf{g}_h^{n-m} - \mathbf{g}^{n-m})\cdot\nabla U_h^{n-m}\|^2 \\ &+ ck\sum_{m=0}^{\nu-1}\|T_h^{\frac{1}{2}}\mathbf{g}^{n-m}\cdot\nabla(U_h^{n-m} - u^{n-m})\|^2 \qquad \theta = \pm\tfrac{1}{2}. \end{aligned} \tag{3.12}$$

From (3.5), (1.3), (3.6), and (1.19), it follows that:

$$\begin{aligned}\|(\mathbf{g}_h^{n-m} - \mathbf{g}^{n-m}) \cdot \nabla U_h^{n-m}\| &\leq cc_\rho \|\xi^{n-m} - \eta^{n-m}\| \|U_h^{n-m}\|_{W^{1,\infty}} \\ &\leq c\bar{c}\|\xi^{n-m}\| + ch^r\|u\|_{r,\infty} \qquad 0 \leq m \leq \nu - 1.\end{aligned} \tag{3.13}$$

Next, set:

$$D^{n-m} \equiv \mathbf{g}^{n-m} \cdot \nabla(U_h^{n-m} - u^{n-m}) \qquad 0 \leq m \leq \nu - 1.$$

By (1.8) and (1.12) with $s = 2$:

$$\|T_h^{\frac{1}{2}} D^{n-m}\|^2 = \|T^{\frac{1}{2}} D^{n-m}\|^2 + ([T_h - T]D^{n-m}, D^{n-m}) \leq c\|D^{n-m}\|_{-1}^2 + ch^2\|D^{n-m}\|^2.$$

Using (1.6) and (1.19):

$$\begin{aligned}\|D^{n-m}\|_{-1} &= \sup_{\varphi \in H_0^1} \frac{|([U_h^{n-m} - u^{n-m}], \nabla \cdot [\varphi \mathbf{g}^{n-m}])|}{\|\varphi\|_1} \leq cc_g\|\xi^{n-m} - \eta^{n-m}\| \\ &\leq c\|\xi^{n-m}\| + ch^r\|u^{n-m}\|_r.\end{aligned}$$

By (1.18), (1.31), and (1.19):

$$h\|D^{n-m}\| \leq cc_g h\|\xi^{n-m} - \eta^{n-m}\|_1 \leq c|||\xi^{n-m}|||_1 + ch^r\|u^{n-m}\|_r.$$

Combining the above inequalities:

$$\|T_h^{\frac{1}{2}} \mathbf{g}^{n-m} \cdot \nabla(U_h^{n-m} - u^{n-m})\| \leq c|||\xi^{n-m}|||_1 + ch^r\|u\|_{r,\infty} \qquad 0 \leq m \leq \nu - 1 \tag{3.14}$$

and (3.7) follows from (3.12) - (3.14). ■

Following (1.31), it is claimed that the relation between h and k can be avoided if $S_h \subset H_0^1$. This can now be seen in the preceeding proof. If the range $\mathcal{R}(T_h^{\frac{1}{2}}) \subset H_0^1$, then using (1.17), an inequality such as (1.8) can be established for $T_h^{\frac{1}{2}}$. Given the latter, it would not be necessary to triangulate with T in order to obtain (3.14). Now if this approach is taken and (1.31) is not assumed, then the consistency result is:

$$\sum_{l=1}^{4} |||\psi_l^n|||_1 \leq ck(h^r + k^\nu + h^2k^{\nu-1})\{\|\partial_t^{\nu+1}u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty} + \|\partial_t u\|_{r,\infty}\}.$$

However, since (1.31) is such a mild condition compared to the requirement that $S_h \subset H_0^1$, the details of the result suggested here are not provided. Instead, (1.9) is established as follows.

Theorem 3.1 *Assume that (1.29), (1.30), and (1.31) hold. Suppose $\{U_h^m\}_{m=0}^{\nu-1}$ are given satisfying:*

$$\max_{0 \leq m \leq \nu-1} |||U_h^m - \omega^m|||_1 \leq c(h^r + k^\nu)\mathcal{N}(u). \tag{3.15}$$

Then provided $h^{-N/2}k^{-1/2}(h^r + k^\nu) + \gamma_1(h)$ is sufficiently small, $\{U_h^n\}_{n=\nu}^{n^*}$ are well-defined by (1.34) and the following holds:

$$\max_{0 \leq n \leq n^*} \|U_h^n - u^n\| \leq c(h^r + k^\nu)\mathcal{N}(u). \tag{3.16}$$

Proof: Set $\theta_0(h,k) \equiv h^{-N/2}(h^r + k^\nu) + \gamma_0(h)$, $\theta_1(h,k) \equiv h^{-N/2}k^{-1/2}(h^r + k^\nu) + \gamma_1(h)$, and let $\bar{c} > c_u$. It is first established that for $\theta_0(h,k)$ and $\theta_1(h,k)$ small enough:

$$\max_{0 \le m \le n^*} \|U_h^m - u^m\|_{L_\infty} \le \rho \qquad \text{and} \qquad \max_{0 \le m \le n^*} \|U_h^m\|_{W^{1,\infty}} \le \bar{c}. \tag{3.17}$$

Note that by (1.14), (3.15), and (1.21), for $\theta_0(h,k)$ small enough:

$$\|U_h^m - u^m\|_{L_\infty} \le ch^{-N/2}\|\xi^m\| + \|\eta^m\|_{L_\infty} \le c\theta_0(h,k) \le \rho \qquad 0 \le m \le \nu - 1.$$

Also, by (1.15), (1.18), (3.15), and (1.22), for $\theta_1(h,k)$ small enough:

$$\|U_h^m\|_{W^{1,\infty}} \le ch^{-N/2}\|\xi^m\|_1 + \|\eta^m\|_{W^{1,\infty}} + c_u \le c\theta_1(h,k) + c_u \le \bar{c} \qquad 0 \le m \le \nu - 1.$$

Now suppose that for each h and k, there exists an $n = n(h,k)$ such that $\nu - 1 \le n \le n^* - 1$ and:

$$\max_{0 \le l \le n} \|U_h^l - u^l\|_{L_\infty} \le \rho, \qquad \max_{0 \le l \le n} \|U_h^l\|_{W^{1,\infty}} \le \bar{c} \tag{3.18}$$

while:

$$\text{either} \quad \|U_h^{n+1} - u^{n+1}\|_{L_\infty} > \rho, \quad \text{or} \quad \|U_h^{n+1}\|_{W^{1,\infty}} > \bar{c}. \tag{3.19}$$

Given (3.18), inequalities (3.2), (3.4), and (3.7) can be combined for the error equation (3.1) to obtain:

$$|||\xi^{l+1}|||_1^2 \le (1 + \tilde{c}k)|||\xi^l|||_1^2 + c_1 k[(h^r + k^\nu)\mathcal{N}(u)]^2 + c_2 k \sum_{m=0}^{\nu-1} |||\xi^{l-m}|||_1^2 \qquad \nu - 1 \le l \le n.$$

After summing this over $\nu - 1 \le l \le n$ and applying the discrete Gronwall Lemma to the result, it follows that:

$$|||\xi^{n+1}|||_1^2 \le (1 + c_3 k)^{n-\nu+1}\{|||\xi^{\nu-1}|||_1^2 + c_1 t^*[(h^r + k^\nu)\mathcal{N}(u)]^2 + c_3 k \sum_{m=0}^{\nu-1} |||\xi^m|||_1^2\} \tag{3.20}$$

for $c_3 \equiv \tilde{c} + c_2\nu \ge 0$. Also, the exponential dependence on t^* can be eliminated if $c_3 < 0$ [13]. Now by (1.14), (3.20), (3.15), and (1.21), for $\theta_0(h,k)$ small enough:

$$\|U_h^{n+1} - u^{n+1}\|_{L_\infty} \le ch^{-N/2}\|\xi^{n+1}\| + \|\eta^{n+1}\|_{L_\infty} \le c\theta_0(h,k) \le \rho.$$

Also, by (1.15), (1.18), (3.20), (3.15), and (1.22), for $\theta_1(h,k)$ small enough:

$$\|U_h^{n+1}\|_{W^{1,\infty}} \le ch^{-N/2}\|\xi^{n+1}\|_1 + \|\eta^{n+1}\|_{W^{1,\infty}} + c_u \le c\theta_1(h,k) + c_u \le \bar{c}.$$

This contradicts (3.19), and hence, (3.17) is established. In fact, (3.16) follows from (3.15) and (3.20) after using (1.19). ■

See [13] for a proof of the following.

Theorem 3.2 *Assume that (1.29), (1.30), and (1.31) hold. Then if* $h^{-N/2}k^{-1/2}(h^r + k^{3/2}) + \gamma_1(h)$ *is sufficiently small,* $\{U_h^{n,i}\}_{0 \le n \le min(i,\nu-1)}^{1 \le i \le 2\nu-1}$ *are well-defined by (1.37) and (1.39), and the following holds:*

$$\max_{0 \le n \le min(i,\nu-1)} |||U_h^{n,i} - \omega^n|||_1 \le c(h^r + k^{i/2+1})\mathcal{N}(u) \qquad 1 \le i \le 2\nu - 1.$$

Therefore, (3.15) follows with $U_h^n \equiv U_h^{n,2(\nu-1)}$, $0 \le n \le \nu - 1$. ■

4 General Semilinearities.

In this section, the principal scheme (1.34) is analyzed for the approximation of the solution to (1.1) in case (1.1.iii), and (1.10) and (1.9) are established. For this, the basic structure of previous sections is followed and in fact, several of the estimates of section 3 are readily adapted. First, the error equation (3.1) serves here but with:

$$f^n \equiv f(\mathbf{x}, t^n, u^n, \nabla u^n), \qquad f_h^n \equiv f(\mathbf{x}, t^n, U_h^n, \nabla U_h^n).$$

Stability is now established in the norm:

$$|||\chi|||_2 \equiv \{(L_h\chi, \chi) + k(L_h^2\chi, \chi)\}^{\frac{1}{2}} \qquad \chi \in S_h.$$

Proposition 4.1 *Suppose that (1.28) is satisfied. Then, for k small enough, there is a constant $\tilde{c} < 0$ such that:*

$$|||\xi^{n+1}|||_2^2 \leq (1 + \tilde{c}k)|||\xi^n|||_2^2 + ck^{-1}\|[I + kL_h](\xi^{n+1} - r_h\xi^n)\|^2. \tag{4.1}$$

Proof: With $c_\epsilon \equiv 1 + \epsilon$ and $\chi^n \equiv [L_h + kL_h^2]^{\frac{1}{2}}\xi^n$, from (3.3) it follows that:

$$\begin{aligned} ([L_h + c_\epsilon kL_h^2]r_h\xi^n, \xi^{n+1}) &= (r_\epsilon(kL_h)r_h\chi^n, \chi^{n+1}) \\ &\leq \tfrac{1}{2}(1 + \tilde{c}k)|||\xi^n|||_2^2 + \tfrac{1}{2}|||\xi^{n+1}|||_2^2. \end{aligned}$$

Also:

$$([L_h + c_\epsilon kL_h^2](\xi^{n+1} - r_h\xi^n), \xi^{n+1}) \leq ck^{-1}\|[I + kL_h](\xi^{n+1} - r_h\xi^n)\|^2 + \epsilon k|||\xi^{n+1}|||_2^2$$

and (4.1) follows after summing these inequalities. ■

Now, the new terms of (4.1) must be treated differently.

Proposition 4.2 *Assume that (1.31) holds. Then the terms $\{\psi_l^n\}_{l=1}^4$ satisfy:*

$$\sum_{l=1}^{4}\|[I + kL_h]\psi_l^n\| \leq ck(h^r + k^\nu)\{\|\partial_t^{\nu+1}u\|_{0,\infty} + \|\partial_t^\nu u\|_{2,\infty} + \|\partial_t u\|_{r,\infty}\}. \tag{4.2}$$

Also, if $\{U_h^{n-m}\}_{m=0}^{\nu-1}$ are given satistying:

$$\max_{0\leq m\leq \nu-1}\|U_h^{n-m} - u^{n-m}\|_{W^{1,\infty}} \leq \rho, \tag{4.3}$$

then ϕ^n satisfies:

$$\|[I + kL_h]\phi^n\| \leq ckh^{r-1}\|u\|_{r,\infty} + ck\sum_{m=0}^{\nu-1}|||\xi^{n-m}|||_2. \tag{4.4}$$

Proof: For (4.2), inequalities (3.8), (3.9), and (3.10) are readily extended by using $\theta = 1$ in (2.1). Then, with $\psi_4^n = P_E H$ as in Proposition 3.2:

$$\|[I + kL_h]\psi_4^n\| \leq \|\psi_4^n\| + k\|P_0 LH\| \leq \|(P_E - I)H\| + \|H\| + k\|LH\|$$

and the remaining component of (4.2) follows as with (3.11). Now, by (2.1):

$$\begin{aligned} \|(kL_h)^\theta\phi^n\| &\leq ck\sum_{m=0}^{\nu-1}\|f(U_h^{n-m}, \nabla U_h^{n-m}) - f(u^{n-m}, \nabla U_h^{n-m}\| \\ &+ ck\sum_{m=0}^{\nu-1}\sum_{j=1}^{N}\|f(u^{n-m}, V_{j-1}^{n-m}) - f(u^{n-m}, V_j^{n-m})\| \qquad \theta = 0, 1 \end{aligned}$$

where $V_0^n \equiv \nabla U_h^n$, $V_N^n \equiv \nabla u^n$ and:

$$V_j^n \equiv \langle \partial_{x_1} u^n, \ldots, \partial_{x_j} u^n, \partial_{x_{j+1}} U_h^n, \ldots, \partial_{x_N} U_h^n \rangle^T \qquad 1 \leq j \leq N-1.$$

Using (4.3), (1.4), (1.19), and (1.18):

$$\begin{aligned} \|f(U_h^{n-m}, \nabla U_h^{n-m}) - f(u^{n-m}, \nabla U_h^{n-m})\| &\leq cc_\rho \|\xi^{n-m} - \eta^{n-m}\| \\ &\leq c|||\xi^{n-m}|||_2 + ch^r \|u\|_{r,\infty} \qquad 0 \leq m \leq \nu - 1. \end{aligned}$$

Similarly:

$$\begin{aligned} \|f(u^{n-m}, V_{j-1}^{n-m}) - f(u^{n-m}, V_j^{n-m})\| &\leq cc_\rho \|\partial_{x_j}(\xi^{n-m} - \eta^{n-m})\| \\ &\leq c|||\xi^{n-m}|||_2 + ch^{r-1} \|u\|_{r,\infty} \\ & \qquad 1 \leq j \leq N, \quad 0 \leq m \leq \nu - 1. \end{aligned}$$

and (3.7) follows after combining the last three inequalities. ■

The groundwork for an H^1 estimate is now complete. For an L_2 estimate, the natural impulse is to press the details surrounding (3.14) for a generalization to the case (1.1.iii). In search of an analogue to the H^{-1} estimate prior to (3.14), it is tempting to suppose the existence of constants ρ, $c_\rho > 0$ such that for $0 \leq t \leq t^*$, the following are satisfied:

$$\begin{cases} \forall U_1, U_2 \in W^{1,\infty} \quad \text{satisfying} \quad \max_{m=1,2;1\leq i\leq N} \|\partial_{x_i}(U_m - u)\|_{L_\infty} \leq \rho \\ \\ \qquad \|f(t, u(t), \nabla U_2) - f(t, u(t), \nabla U_1)\|_{-1} \leq c_\rho \|U_2 - U_1\| \end{cases}$$

and:

$$\begin{cases} \forall v \in W^{2,\infty} \quad \text{satisfying} \quad \max_{1\leq i\leq N} \|\partial_{x_i}(v - u)\|_{L_\infty} \leq \rho \\ \\ \qquad f(\mathbf{x}, t, u(\mathbf{x}, t), \nabla v(\mathbf{x})) \in W^{1,\infty}. \end{cases}$$

However, it is shown in [14] that these conditions are actually equivalent to the following:

$$\begin{cases} \exists \ \{f_m(\mathbf{x}, t, u(\mathbf{x}, t))\}_{m=0}^N \subset W^{1,\infty}, \quad \mathbf{f} \equiv \langle f_1, f_2, \ldots, f_N \rangle^T \\ \\ \text{such that} \quad \forall w \in W^{1,\infty} \quad \text{satisfying} \quad \max_{1\leq i\leq N} \|\partial_{x_i}(w - u)\|_{L_\infty} \leq \rho \\ \\ f(\mathbf{x}, t, u(\mathbf{x}, t), \nabla w(\mathbf{x})) = f_0(\mathbf{x}, t, u(\mathbf{x}, t)) + \mathbf{f}(\mathbf{x}, t, u(\mathbf{x}, t)) \cdot \nabla w(\mathbf{x}). \end{cases}$$

In fact, this is at the heart of the division (1.1.i) - (1.1.iii). Nevertheless, it is clear that a new approach is required for the term ϕ^n before an optimal L_2 estimate can be proved.

Proposition 4.3 *Assume that (1.31) holds and suppose* $\{U_h^{n-m}\}_{m=0}^{\nu-1}$ *are given satisfying:*

$$\max_{0\leq m\leq \nu-1} \|U_h^{n-m} - u^{n-m}\|_{W^{1,\infty}} \leq \rho, \tag{4.5}$$

Then with $\epsilon \in (0,1)$, $p_{N,\epsilon} \equiv max\{0, \frac{N}{2} - 1 + \epsilon\}$, ϕ^n *satisfies:*

$$\begin{aligned} \|(kL_h)^{-\frac{1}{2}}\phi^n\|^2 + \|(kL_h)^{\frac{1}{2}}\phi^n\|^2 &\leq ck(h^r\|u\|_{r,\infty})^2 + ck\sum_{m=0}^{\nu-1} |||\xi^{n-m}|||_1^2 \\ &+ c(\epsilon)kh^{-2p_{N,\epsilon}} \sum_{m=0}^{\nu-1} \|U_h^{n-m} - u^{n-m}\|_1^4. \end{aligned} \tag{4.6}$$

Proof: By (1.17) and (2.1):

$$\begin{aligned} \|(kL_h)^{\theta}\phi^n\|^2 &\leq ck\sum_{m=0}^{\nu-1}\|f(U_h^{n-m},\nabla U_h^{n-m}) - f(u^{n-m},\nabla U_h^{n-m})\|^2 \\ &+ ck\sum_{m=0}^{\nu-1}\|T_h^{\frac{1}{2}}[f(u^{n-m},\nabla U_h^{n-m}) - f(u^{n-m},\nabla u^{n-m})]\|^2 \qquad \theta = \pm\tfrac{1}{2}. \end{aligned} \tag{4.7}$$

From (4.5), (1.4), and (1.19), it follows that:

$$\begin{aligned} \|f(U_h^{n-m},\nabla U_h^{n-m}) - f(u^{n-m},\nabla U_h^{n-m})\| &\leq cc_{\rho}\|\xi^{n-m} - \eta^{n-m}\| \\ &\leq c\|\xi^{n-m}\| + ch^r\|u\|_{r,\infty} \qquad 0 \leq m \leq \nu - 1. \end{aligned} \tag{4.8}$$

Next, combining (4.5) and (1.5), $\{D_j^{n-m}\}_{1\leq j\leq N}^{0\leq m\leq \nu-1}$ and $\{E_{ij}^{n-m}\}_{1\leq i,j\leq N}^{0\leq m\leq\nu-1}$ are well-defined by the following:

$$\begin{aligned} f(u^{n-m},\nabla U_h^{n-m}) - f(u^{n-m},\nabla u^{n-m}) &= \sum_{j=1}^{N}\partial_{x_j}(U_h^{n-m}-u^{n-m})\partial_{d_j}f(u^{n-m},\nabla u^{n-m}) + \\ \sum_{i,j=1}^{N}\partial_{x_i}(U_h^{n-m}-u^{n-m})\partial_{x_j}(U_h^{n-m}-u^{n-m})&\int_0^1(1-s)\partial^2_{d_id_j}f(u^{n-m},\nabla[u^{n-m}+s(U_h^{n-m}-u^{n-m})]ds \\ &\equiv \sum_{j=1}^{N}D_j^{n-m} + \sum_{i,j=1}^{N}E_{ij}^{n-m} \qquad 0 \leq m \leq \nu - 1. \end{aligned} \tag{4.9}$$

Now, by (1.8) and (1.12) with $s = 2$:

$$\begin{aligned} \|T_h^{\frac{1}{2}}D_j^{n-m}\|^2 &\leq \|T^{\frac{1}{2}}D_j^{n-m}\|^2 + ([T_h - T]D_j^{n-m}, D_j^{n-m}) \\ &\leq c\|D_j^{n-m}\|_{-1}^2 + ch^2\|D_j^{n-m}\|^2 \qquad 1 \leq j \leq N, \quad 0 \leq m \leq \nu - 1. \end{aligned}$$

The first of these two terms is estimated as follows using (1.7), and (1.19):

$$\begin{aligned} \|D_j^{n-m}\|_{-1} &= \sup_{\varphi\in H_0^1}\frac{|([U_h^{n-m} - u^{n-m}], \partial_{x_j}[\varphi\partial_{d_j}f^{n-m}])|}{\|\varphi\|_1} \\ &\leq cc_f\|\xi^{n-m} - \eta^{n-m}\| \leq c|\!|\!|\xi^{n-m}|\!|\!|_1 + ch^r\|u^{n-m}\|_r \\ &\qquad 1 \leq j \leq N, \quad 0 \leq m \leq \nu - 1. \end{aligned}$$

For the second term above, with (1.7), (1.18), (1.31), and (1.19) it follows that:

$$\begin{aligned} h\|D_j^{n-m}\| &\leq cc_fh\|\xi^{n-m} - \eta^{n-m}\|_1 \leq c|\!|\!|\xi^{n-m}|\!|\!|_1 + ch^r\|u^{n-m}\|_r \\ &\qquad 1 \leq j \leq N, \quad 0 \leq m \leq \nu - 1. \end{aligned}$$

Combining the last three inequalities:

$$\|T_h^{\frac{1}{2}}\sum_{j=1}^{N}D_j^{n-m}\| \leq c|||\xi^{n-m}|||_1 + ch^r\|u\|_{r,\infty} \qquad 0 \leq m \leq \nu - 1. \tag{4.10}$$

Next, using (4.5), and (1.5):

$$\begin{aligned}\|T_h^{\frac{1}{2}}E_{ij}^{n-m}\| &= \sup_{\varphi\in L_2}\frac{|(E_{ij}^{n-m}, T_h^{\frac{1}{2}}\varphi)|}{\|\varphi\|} \leq cc_\rho\|U_h^{n-m} - u^{n-m}\|_1^2 \sup_{\varphi\in L_2}\frac{\|T_h^{\frac{1}{2}}\varphi\|_{L_\infty}}{\|\varphi\|} \\ &\qquad 1 \leq i,j \leq N, \quad 0 \leq m \leq \nu - 1.\end{aligned} \tag{4.11}$$

By the Sobolev Imbedding Theorem (See Adams [1]), since $S_h \subset W^{1,\infty}$, for $\epsilon \in (0,1)$:

$$\|T_h^{\frac{1}{2}}\varphi\|_{L_\infty} = \sup_{\mathbf{x}\in\Omega}|T_h^{\frac{1}{2}}\varphi(\mathbf{x})| \leq c(\epsilon)\|T_h^{\frac{1}{2}}\varphi\|_{W^{1,N(1-\epsilon)^{-1}}} \qquad \forall\varphi \in L_2.$$

In case $N = 1$, with $\epsilon = \frac{1}{2}$ this is sufficient for what follows. For $N \geq 2$, note that by (1.16):

$$\|T_h^{\frac{1}{2}}\varphi\|_{W^{1,N(1-\epsilon)^{-1}}} \leq ch^{-N/2+1-\epsilon}\|T_h^{\frac{1}{2}}\varphi\|_1 \qquad \forall\varphi \in L_2.$$

In any case, from (1.17) and the last two inequalities, it follows that:

$$\|T_h^{\frac{1}{2}}\varphi\|_{L_\infty} \leq c(\epsilon)h^{-p_{N,\epsilon}}\|\varphi\| \qquad \forall\varphi \in L_2. \tag{4.12}$$

Combining (4.11) and (4.12):

$$\|T_h^{\frac{1}{2}}\sum_{i,j=1}^{N}E_{ij}^{n-m}\| \leq c(\epsilon)h^{-p_{N,\epsilon}}\|U_h^{n-m} - u^{n-m}\|_1^2 \qquad 0 \leq m \leq \nu - 1. \tag{4.13}$$

Then, combining (4.10) and (4.13) for (4.9):

$$\begin{aligned}\|T_h^{\frac{1}{2}}[f(u^{n-m}, \nabla U_h^{n-m}) - f(u^{n-m}, \nabla u^n)]\| &\leq ch^r\|u\|_{r,\infty} + c|||\xi^{n-m}|||_1 \\ &+ c(\epsilon)h^{-p_{N,\epsilon}}\|U_h^{n-m} - u^{n-m}\|_1^2 \\ &\qquad 0 \leq m \leq \nu - 1.\end{aligned} \tag{4.14}$$

Now, (4.6) follows after combining (4.14) and (4.8) for (4.7). ■

Note that the remarks made prior to Theorem 3.1 apply here as well. Nevertheless, the convergence results (1.10) and (1.9) are now established as follows.

Theorem 4.1 *Assume that (1.29), (1.30), and (1.31) hold. Suppose $\{U_h^m\}_{m=0}^{\nu-1}$ are given satisfying:*

$$\max_{0\leq m\leq\nu-1}|||U_h^m - \omega^m|||_2 \leq c(h^{r-1} + k^\nu)\mathcal{N}(u). \tag{4.15}$$

Then provided $h^{-N/2}(h^{r-1}+k^\nu)+\gamma_1(h)$ is sufficiently small, $\{U_h^n\}_{n=\nu}^{n^*}$ are well-defined by (1.34) and the following holds:

$$\max_{0\leq n\leq n^*}\|U_h^n - u^n\|_1 \leq c(h^{r-1} + k^\nu)\mathcal{N}(u). \tag{4.16}$$

If $\{U_h^m\}_{m=0}^{\nu-1}$ also satisfy:

$$\max_{0\le m\le \nu-1} |||U_h^m - \omega^m|||_1 \le c(h^r + k^\nu)\mathcal{N}(u). \tag{4.17}$$

then:

$$\max_{0\le n\le n^*} \|U_h^n - u^n\| \le c(h^r + k^\nu)\mathcal{N}(u). \tag{4.18}$$

Proof: Set $\theta(h,k) \equiv h^{-N/2}(h^{r-1} + k^\nu) + \gamma_1(h)$. It is first established that for $\theta(h,k)$ small enough:

$$\max_{0\le m\le n^*} \|U_h^m - u^m\|_{W^{1,\infty}} \le \rho. \tag{4.19}$$

Note that by (1.15), (1.18), (4.15), and (1.22), for $\theta(h,k)$ small enough:

$$\|U_h^m - u^m\|_{W^{1,\infty}} \le ch^{-N/2}\|\xi^m\|_1 + \|\eta^m\|_{W^{1,\infty}} \le c\theta(h,k) \le \rho \qquad 0 \le m \le \nu - 1.$$

Now suppose that for each h and k, there exists an $n = n(h,k)$ such that $\nu - 1 \le n \le n^* - 1$ and:

$$\max_{0\le l\le n} \|U_h^l - u^l\|_{W^{1,\infty}} \le \rho, \tag{4.20}$$

while:

$$\|U_h^{n+1} - u^{n+1}\|_{W^{1,\infty}} > \rho. \tag{4.21}$$

Given (4.20), inequalities (4.1), (4.2), and (4.4) can be combined to obtain:

$$|||\xi^{l+1}|||_2^2 \le (1 + \tilde{c}k)|||\xi^l|||_2^2 + c_1 k[(h^{r-1} + k^\nu)\mathcal{N}(u)]^2 + c_2 k \sum_{m=0}^{\nu-1} |||\xi^{l-m}|||_2^2 \qquad \nu - 1 \le l \le n.$$

After summing this over $\nu - 1 \le l \le n$ and applying the discrete Gronwall Lemma to the result, it follow that:

$$|||\xi^{n+1}|||_2^2 \le (1 + c_3 k)^{n-\nu+1}\{|||\xi^{\nu-1}|||_2^2 + c_1 t^*[(h^{r-1} + k^\nu)\mathcal{N}(u)]^2 + c_3 k \sum_{m=0}^{\nu-1} |||\xi^m|||_2^2\} \tag{4.22}$$

for $c_3 \equiv \tilde{c} + c_2\nu \ge 0$. Also, the exponential dependence on t^* can be eliminated if $c_3 \le 0$ [13]. Now, by (1.15), (1.18), (4.22), (4.15), and (1.22), for $\theta(h,k)$ small enough:

$$\|U_h^{n+1} - u^{n+1}\|_{W^{1,\infty}} \le ch^{-N/2}\|\xi^{n+1}\|_1 + \|\eta^{n+1}\|_{W^{1,\infty}} \le c\theta(h,k) \le \rho.$$

Hence, (4.21) is contradicted and (4.19) is established. In fact, (4.16) follows from (4.22) and (4.15) after using (1.19).

Now, if (4.17) holds, (4.18) is obtained as follows. Given (4.19), inequalities (3.2), (4.2), and (4.6) can be combined to obtain:

$$|||\xi^{l+1}|||_1^2 \le (1 + \tilde{c}k)|||\xi^l|||_1^2 + c_4 k[(h^r + k^\nu)\mathcal{N}(u)]^2 + c_5 k \sum_{m=0}^{\nu-1} |||\xi^{l-m}|||_1^2 \qquad \nu - 1 \le l \le n^* - 1$$

since for small enough ϵ and $\theta(h,k)$, $\quad r - 2 - p_{N,\epsilon} \ge 0 \quad$ and:

$$\{c(h^r + k^\nu)\mathcal{N}(u) + c(\epsilon)h^{-p_{N,\epsilon}}[(h^{r-1} + k^\nu)\mathcal{N}(u)]^2\}^2 \le c[h^{r-2-p_{N,\epsilon}} + \theta(h,k)]^2[(h^r + k^\nu)\mathcal{N}(u)]^2$$

$$\le c_4[(h^r + k^\nu)\mathcal{N}(u)]^2.$$

After summing over $\nu - 1 \leq l \leq n \leq n^* - 1$ and applying the discrete Gronwall Lemma to the result, it follows that:

$$|||\xi^{n+1}|||_1^2 \leq (1 + c_6 k)^{n-\nu+1} \{|||\xi^{\nu-1}|||_1^2 + c_4 t^* [(h^r + k^\nu)\mathcal{N}(u)]^2 + c_6 k \sum_{m=0}^{\nu-1} |||\xi^m|||_1^2\} \tag{4.23}$$

for $c_6 \equiv \tilde{c} + c_4 \nu \geq 0$. Again, the exponential dependence on t^* can be eliminated if $c_6 < 0$ [13]. Finally, (4.18) follows from (4.23) and (4.17) after using (1.19). ■

See [13] for a proof of the next Theorem. It requires the inequality:

$$\|L_h^{\frac{1}{2}} \chi\| \leq c h^{-1} \|\chi\| \qquad \forall \chi \in S_h \tag{4.24}$$

which depends on inverse properties such as (1.14) - (1.16).

Theorem 4.2 *Assume that (1.29), (1.30), (1.31) and (4.24) are satisfied. Then provided* $h^{-N/2}(h^{r-1} + k^{3/2}) + \gamma_1(h)$ *is sufficiently small,* $\{U_h^{n,i}\}_{0 \leq n \leq min(i,\nu-1)}^{1 \leq i \leq 2\nu-1}$ *are well-defined by (1.37) and (1.39) and the following holds:*

$$\max_{0 \leq n \leq min(i,\nu-1)} |||U_h^{n,i} - \omega^n|||_2 \leq c(h^{r-1} + k^{i/2+1})\mathcal{N}(u) \qquad 1 \leq i \leq 2\nu - 1.$$

Therefore, (4.15) follows with $U_h^n \equiv U_h^{n,2(\nu-1)}$, $0 \leq n \leq \nu - 1$. *Furthermore:*

$$\max_{0 \leq n \leq min(i,\nu-1)} |||U_h^{n,i} - \omega^n|||_1 \leq c(h^r + k^{(i+1)/2})\mathcal{N}(u) \qquad 1 \leq i \leq 2\nu - 1$$

and (4.17) follows with $U_h^n \equiv U_h^{n,2\nu-1}$, $0 \leq n \leq \nu - 1$.

5 Computational Aspects.

Note from (1.34) and (1.37), the need of an efficient method for solving linear equations of the form:

$$[I + kA\mathcal{L}_h]\Psi = \Phi \qquad \Psi, \Phi \in \mathbf{S}_h.$$

In connection with parallel implementation, it can be seen below that the preferred methods are those for which the eigenvalues of A are distinct. So, let these be called *multiply implicit* (MIRK) methods.

First, consider the use of IRKM's which have high order with respect to q, such as the *Gauss-Legendre* ($\nu = 2q$) or the *Radau* ($\nu = 2q - 1$) methods (See e. g., Dekker and Verwer [8]). These are A_0-stable, and the latter are strongly A_0-stable while the former are not. Also, though each class of methods satisfies (1.30), for neither is $\sigma(A) \subset \mathbf{R}$. Nevertheless, these are (complex) MIRK's, for which A can always be transformed to quasidiagonal form:

$$A = S^{-1} \Lambda S$$

where for some m, $1 \leq m < q$:

$$\Lambda = \operatorname*{diag}_{1 \leq i \leq m} \{\Lambda_i\},$$

and either:

$$\Lambda_i = \lambda_i > 0 \quad \text{or} \quad \Lambda_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix} \qquad \alpha_i, \beta_i > 0 \qquad 1 \leq i \leq m.$$

Then, the linear system can be written as:

$$[I + k\Lambda\mathcal{L}_h](S\Psi) = (S\Phi)$$

which decouples to equations of the form:

$$[I + k\lambda L_h]\psi = \phi \qquad \psi, \phi \in S_h$$

and:

$$\begin{bmatrix} I + k\alpha L_h & -k\beta L_h \\ k\beta L_h & I + k\alpha L_h \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \qquad \psi_i, \phi_i \in S_h, \quad i = 1, 2.$$

Note that the subordinate equations can in principle, be solved simultaneously. Further, the solutions ψ_1 and ψ_2 for the 2×2 system can be computed in parallel according to:

$$[I + 2\alpha k L_h^n + (\alpha^2 + \beta^2)(kL_h^n)^2]\psi_1 = [I + k\alpha L_h^n]\phi_1 + k\beta L_h^n \phi_2 \equiv \chi_1$$

and:

$$[I + 2\alpha k L_h^n + (\alpha^2 + \beta^2)(kL_h^n)^2]\psi_2 = [I + k\alpha L_h^n]\phi_2 - k\beta L_h^n \phi_1 \equiv \chi_2.$$

Following Baker, Bramble and Thomée [2], since for $x \in \mathbf{R}$:

$$\frac{1}{1 + 2\alpha x + (\alpha^2 + \beta^2)x^2} = \Re\left[\frac{z_1}{1 + z_2 x}\right] \qquad z_1 \equiv 1 - \alpha\beta^{-1} i, \quad z_2 \equiv \alpha + \beta i,$$

complex arithmetic can be used to obtain:

$$\psi_i = \Re\{z_1[I + z_2 k L_h]^{-1}\chi_i\} \qquad i = 1, 2.$$

Next, in spite of the order barrier $\nu \leq q + 1$, consider methods for which A is similar to a matrix of the form:

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ \theta_2 & \lambda_2 & & \\ & \ddots & \ddots & \\ 0 & & \theta_q & \lambda_q \end{bmatrix} \qquad \begin{array}{ll} \lambda_i > 0, & 1 \leq i \leq q \\ \\ \theta_i = 0 \text{ or } 1, & 2 \leq i \leq q \end{array}$$

so that the block system reduces to equations which are linear in L_h:

$$\begin{cases} [I + k\lambda_1 L_h]\,\psi_1 = \phi_1 & \\ [I + k\lambda_i L_h]\,\psi_i = \phi_i - k\theta_i L_h \psi_{i-1}, & 2 \leq i \leq q \end{cases} \qquad \psi_j, \phi_j \in S_h, \quad 1 \leq j \leq q.$$

A class of (real) MIRK's is obtained in case $\lambda_i \neq \lambda_j$, $i \neq j$ and $\theta_i = 0$, $2 \leq i \leq q$. The depth of decoupling these methods allow in the above equations makes them appear, in a parallel environment, more attractive than their complex counterparts above. However, the cost of this advantage is reduced order.

On the other hand, the class for which $\lambda_i = \lambda$, $1 \leq i \leq q$ is said to contain *singly implicit* (SIRK) methods. Since these require the formation and factorization of only a single matrix with the dimension of S_h, SIRK's are preferred if computations are to be performed on a serial machine. A selection from the previous set was made for the example discussed below.

The following problem is of the class considered in Section 2:

$$\begin{cases} \partial_t u = \partial_x^2 u + f(x, t, u) & \text{in} \quad (-1, 1) \times [0, .1] \\ u = 0 & \text{on} \quad \{-1, 1\} \times [0, .1] \\ u(x, 0) = 1 - x^2 & \text{in} \quad (-1, 1) \end{cases}$$

k, h	CPU Time (sec)	L_2 error ($\times 10^9$)	Order
1/50	12	21.4	
1/60	17	9.94	4.20
1/70	23	5.30	4.08
1/80	31	3.09	4.03
1/90	39	1.93	4.01
1/100	45	1.26	4.00

Table 1: Modified method

where:

$$f(x,t,u) = \frac{2-(1-x^2)\log(1+X^2)}{(1+X^2)^t} + \frac{2t(1-5x^2)}{(1+X^2)^{t+1}} - \frac{4x^2t(t+1)(1-x^2)}{(1+X^2)^{t+2}}, \qquad X^2 \equiv 1-(1+x^2)^t u.$$

The solution is given by:

$$u(x,t) = \frac{1-x^2}{(1+x^2)^t}.$$

For the spatial discretization, the Ordinary Galerkin Method was used and S_h was constructed of smooth cubic splines defined on a uniform mesh. For the temporal discretization, the following three-stage *diagonally implicit* (DIRK) method was used:

$$\begin{array}{ccc|c} \gamma & 0 & 0 & \gamma \\ \frac{1}{2}-\gamma & \gamma & 0 & \frac{1}{2} \\ 2\gamma & 1-4\gamma & \gamma & 1-\gamma \\ \hline \beta & 1-2\beta & \beta & \end{array} \qquad \begin{array}{l} \gamma \equiv \frac{1}{2} + \frac{1}{\sqrt{3}}\cos(\frac{\pi}{18}) \\ \\ \beta \equiv [6(1-2\gamma)^2]^{-1}. \end{array}$$

In the sequel, let (1.34), (1.37), and (1.38) be identified as the *modified* method. For comparison, a *classical* method is now introduced. Define $\mathcal{F}^n : \mathbf{L}_2 \to \mathbf{L}_2$ to have components:

$$\mathcal{F}_i^n(\Phi) \equiv f(\mathbf{x}, t^n + k\tau_i, \phi_i) \qquad \Phi \in \mathbf{S}_h.$$

Next, let $\bar{U}_h^{0,0} \equiv eU_h^0 = eP_0u^0$ and after the stages $\{\bar{U}_h^{n-m}\}_{m=1}^{\min(n,\nu)}$ are computed as indicated below, take:

$$\bar{U}_h^{n,0} \equiv \sum_{m=1}^{\min(n,\nu)} (-1)^{m+1} \left(\begin{array}{c} \min(n,\nu) \\ m \end{array} \right) \bar{U}_h^{n-m} \qquad 1 \le n \le n^* - 1.$$

Then define:

$$\bar{U}_h^{n,l} \equiv [I + kA\mathcal{L}_h]^{-1}\{eU_h^n + kA\mathcal{P}_0\mathcal{F}^n(\bar{U}_h^{n,l-1})\} \qquad l \ge 1, \quad 0 \le n \le n^* - 1$$

and:

$$\bar{U}_h^n \equiv \bar{U}_h^{n,\max(\nu+1-n,1)} \qquad 0 \le n \le n^* - 1.$$

Finally, take:

$$U_h^{n+1} = (I - b^T A^{-1} e)U_h^n + b^T A^{-1} \bar{U}_h^n.$$

Now, in addition to the modified and classical methods, let a *hybrid* method be given by (1.34), (1.37), and (1.38), but with $l!\,A^l e$ replaced by $\mathrm{T}^l e$ in (1.32) and (1.35).

These three methods were tested on the ICASE SUN 3/180. In Tables 1 - 3, the L_2 errors:

$$E(k) \equiv E(k,k), \qquad E(h,k) \equiv \|U_h^{n^*} - u^{n^*}\|$$

k, h	CPU Time (sec)	L_2 error ($\times 10^9$)	Order
1/50	51	408.	
1/60	67	245.	2.80
1/70	87	158.	2.84
1/80	112	107.	2.88
1/90	136	76.4	2.91
1/100	168	56.0	2.94

Table 2: Classical method

k, h	CPU Time (sec)	L_2 error ($\times 10^9$)	Order
1/50	12	402.	
1/60	17	244.	2.75
1/70	22	158.	2.82
1/80	29	107.	2.87
1/90	37	76.3	2.91
1/100	46	56.0	2.94

Table 3: Hybrid method

are reported together with estimates of the convergence order obtained according to the formula:

$$\text{Order} \equiv \frac{\log(E(k_2)/E(k_1))}{\log(k_2/k_1)}.$$

As expected, the classical method required much more time because of the additional function evaluations. On the other hand, no rigorous explanation can be offered for the identical accuracy obtained by the classical and hybrid methods. Also, this author is unaware of any proof of the better than second order convergence seen in Tables 2 and 3. In this connection, note that with $Lv \equiv \partial_x^2 v$, $v \in H^2 \cap H_0^1$, the above solution has no time derivatives which are even in the domain of L^2. Nevertheless, only second order convergence is demonstrated for example, in Experiment 7.5.2 of Dekker and Verwer [8], where a stiff nonlinear ordinary differential equation is considered. Further, the modified method has been applied to this problem to give not only fourth order convergence, but accuracy exceeding that reported for any method discussed in the Experiment.

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