Decay Estimate Addendum to:
Revision of the Theory of Tracer Transport and the Convolution Model of DCE-MRI

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Abstract. The convolution model of DCE-MRI is considered in [2] where a revised theory of tracer transport is set forth. To arrive at a model which is purely local to the tissue region being examined, it was assumed in [2] that the selected tissue has only convective and no diffusive coupling with outside regions. Then the arterial input function was introduced as a forcing term in a distributed convection-diffusion system. A consequence of this formulation was that a decay estimate was not forthcoming, which corresponds physically to the natural condition that contrast agent is eventually eliminated from the tissue region. The purpose of this Addendum is to present a more elegant formulation for which a natural exponential decay estimate can be proved.

Keywords: convection, diffusion, perfusion, permeation, tracer, transport, DCE-MRI, convolution, semigroup, decay.

1 Review of the Problem

The problem of modeling dynamic contrast enhanced magnetic resonance imaging (DCE-MRI) is considered in [2] where a revised theory of tracer transport is set forth. According to this technique, a bolus of contrast agent is injected into a patient, and rapid magnetic resonance imaging techniques are used to follow the concentration time course of the contrast agent as illustrated in Figure 1. The local changes in intensity seen in such data are used to quantify the transport properties of the imaged tissue. The transport of contrast agent through an arbitrary tissue region Ω is illustrated diagramatically in Figure 2. Here, C_{AIF} is the *arterial input function* which represents the concentration of contrast agent when entering the tissue region Ω. The *venous output function* is C_{VOF} which represents the concentration of contrast agent when exiting Ω. The average concentration in the tissue region is denoted by C_{T}. Since C_{AIF} and C_{T} can be estimated from imaging while C_{VOF} cannot, the modeling goal is to relate C_{AIF} and C_{T} in such a way that measurements of these quantities permit transport properties to be estimated.

The distributed concentration $C = C(x,t)$ in Ω is formulated in [2] according to the following convection-diffusion partial differential equation with imposed initial and boundary values:

$$\left\{ \begin{array}{rcll} \partial_t C + \nabla \cdot [F\boldsymbol{v}C] & = & \nabla \cdot [D(\boldsymbol{v})\nabla C], & \Omega \times (0,T] \\ 0 & = & \boldsymbol{n} \cdot [D(\boldsymbol{v})\nabla C], & \Sigma \times (0,T] \\ C & = & C_{\mathrm{AIF}}, & \Gamma \times (0,T] \\ C & = & C_0, & \Omega \times \{t=0\} \end{array} \right. \tag{1.1}$$

Here, $\boldsymbol{v}$ is a unit vector field, $|\boldsymbol{v}(\boldsymbol{x})| = 1$, with the same orientation as convection, and F is the fluid velocity. Also, $D(\boldsymbol{v}) = d\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}} + p[I - \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}]$ is a diffusivity tensor which includes a convection oriented diffusivity d and a permeation oriented diffusivity p. The outwardly directed unit normal vector at the boundary $\partial\Omega$ is denoted by $\boldsymbol{n}$. Thus $\Gamma = \{\boldsymbol{x} \in \partial\Omega : F\boldsymbol{v} \cdot \boldsymbol{n} < 0\}$ is the inflow portion of the boundary and the rest of the boundary is denoted by $\Sigma = \partial\Omega \backslash \Gamma$.

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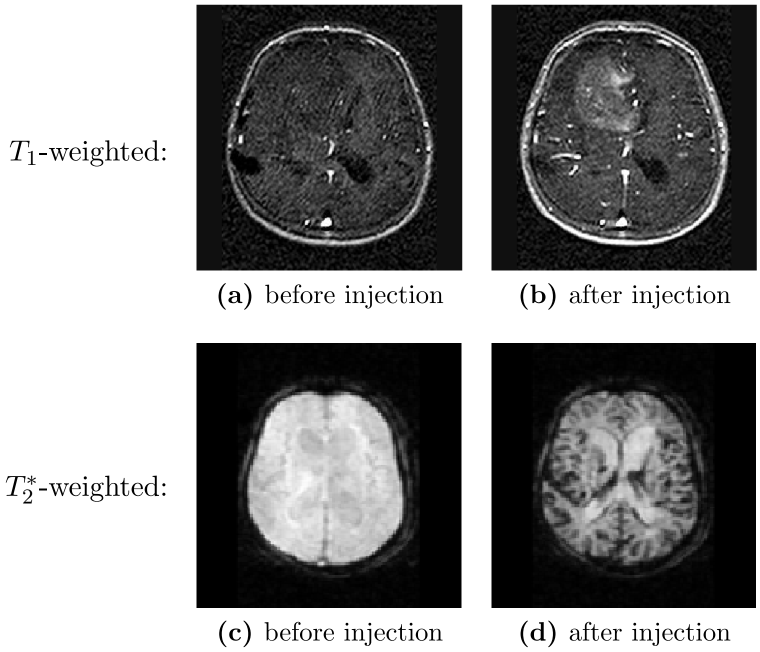

Figure 1: Magnetic resonance images taken from a series of images measured during the injection of a Gadolinium-DTPA based contrast agent. From left to right the images were taken, respectively, before and after the appearance of the contrast agent. The first row shows T_1-weighted images for which the appearance of contrast agent has caused a local elevation in the otherwise rather uniform intensity. The second row shows T_2^*-weighted images for which the contrast agent causes a local reduction in intensity.

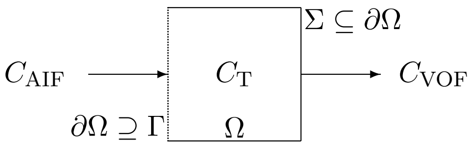

Figure 2: Contrast agent enters the tissue region Ω with concentration C_{AIF} and leaves with concentration C_{VOF}. The average concentration within the tissue region is C_{T}. The inflow boundary is Γ and the outflow boundary is a subset of Σ.

As seen in [2], systems such as (1.1) can be integrated to arrive at a convolution formulation:

$$C_{\text{T}}(t) = \int_0^t K(t-\tau)C_{\text{AIF}}(\tau)d\tau \tag{1.2}$$

where K is a convolution kernel depending upon the transport parameters. To arrive at such a convolution model, which is purely local to the selected tissue and isolated from its surroundings, it was assumed in [2] that Ω has only convective and no diffusive coupling with outside regions. Specifically, the boundary condition in (1.1) on Σ was assumed to hold on the whole of $\partial\Omega$ leaving $\Gamma = \emptyset$. Then C_{AIF} was introduced as a forcing term in the partial differential equation as a parallel to a compartmental formulation. A consequence of this formulation was that a decay estimate was not forthcoming, which corresponds physically to the natural condition that contrast agent is eventually eliminated from Ω. The purpose of this Addendum is to present a more elegant formulation in which (1.1) is used directly to establish the characterization (1.2) for which an exponential decay estimate can be proved.

2 Statement of the Result

Here (1.1) will be written as a Cauchy problem,

$$\begin{cases} X'(t) &= -AX(t) + U(t), \quad t > 0 \\ X(0) &= X_0 \end{cases} \tag{2.1}$$

which is solved formally with $T(t) = \exp[-At]$ and:

$$X(t) = T(t)X_0 + \int_0^t T(t-s)U(s)ds \tag{2.2}$$

For the following note that $H^m(\Omega)$ is the Sobolev space of Lebesgue measurable functions whose weak derivatives up to order m are in the space $L^2(\Omega)$ of Lebesgue square integrable functions. The space of essentially bounded Lebesgue measurable functions is denoted by $L^\infty(\Omega)$. Also, the space of continuous mappings from U into V is denoted by $\mathcal{C}(U, V)$ and by $\mathcal{C}(U)$ in case $V = \boldsymbol{R}$. The space of bounded linear operators from U into V is denoted by $\mathcal{L}(U, V)$ and by $\mathcal{L}(U)$ in case $V = U$. The norm for the Banach space B is denoted by $\|\cdot\|_B$ and the inner product on the Hilbert space H is denoted by $\langle \cdot, \cdot \rangle_H$.

To transform (1.1) into the form (2.1), define first the Hilbert space:

$$H^1_\Gamma(\Omega) = \{X \in H^1(\Omega) : X = 0 \text{ on } \Gamma\} \tag{2.3}$$

which as a closed subspace of $H^1(\Omega)$ may be equipped with the usual $H^1(\Omega)$ norm. Then by integrating the convection-diffusion terms in (1.1) by parts define the following bilinear form on $H^1_\Gamma(\Omega) \times H^1_\Gamma(\Omega)$:

$$a(X, Y) = \langle D(\boldsymbol{v})\nabla X, \nabla Y \rangle_{L^2(\Omega)} + \langle F\boldsymbol{v} \cdot \nabla X, Y \rangle_{L^2(\Omega)} \tag{2.4}$$

As consequence of Theorem 4 below is that there exists a unique operator A defined by:

$$a(X, Y) = \langle AX, Y \rangle_{L^2(\Omega)}, \quad X \in D(A), \quad Y \in H^1_\Gamma(\Omega) \tag{2.5}$$

where

$$D(A) = \{X \in H^1_\Gamma(\Omega) : |a(X, Y)| \le c\|Y\|_{L^2(\Omega)}, \forall Y \in H^1_\Gamma(\Omega)\} \tag{2.6}$$

It is assumed, in the standard way [2], that the boundary condition in (1.1) is spatially invariant, $C_{\text{AIF}} = C_{\text{AIF}}(t)$, and sufficiently regular. Specifically,

$$C_{\text{AIF}} \in \mathcal{C}^{1,\gamma}([0, \infty)), \quad \gamma \in (0, 1] \tag{2.7}$$

To extend the boundary condition into the interior of the tissue, define:

$$\bar{C}(\boldsymbol{x}, t) = C_{\text{AIF}}(t) I_\Omega(\boldsymbol{x}), \quad I_\Omega(\boldsymbol{x}) = 1, \quad \bar{C} \in \mathcal{C}^{1,\gamma}([0, \infty), L^2(\Omega)) \tag{2.8}$$

and

$$U = -\bar{C}' \in \mathcal{C}^{0,\gamma}([0, \infty), L^2(\Omega)). \tag{2.9}$$

With $\bar{C}$ given according to (2.8), a solution C to (1.1) is transformed to a function satisfying homogeneous Dirichlet boundary conditions at Γ according to:

$$X = C - \bar{C} \tag{2.10}$$

which satisfies (2.1) with the initial condition,

$$X_0 = C_0 - \bar{C}(0). \tag{2.11}$$

The technical assumptions required of the transport coefficients are now stated explicitly. It is assumed that the diffusivity tensor is uniformly coercive in Ω:

$$\delta|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi} \cdot [D(\boldsymbol{v})\boldsymbol{\xi}], \quad \forall \boldsymbol{\xi} \in \boldsymbol{R}^3 \tag{2.12}$$

Also, the coefficients are assumed to be temporally invariant, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}), F = F(\boldsymbol{x}), d = d(\boldsymbol{x}), p = p(\boldsymbol{x})$, bounded, $F, d, p \in L^\infty(\Omega)$ and sufficiently smooth. Further, it is assumed that $\partial\Omega$ is smooth and that the fluid flow is balanced in every subset with smooth boundary:

$$\int_{\partial\Omega_0} [F\boldsymbol{v} \cdot \boldsymbol{n}] d\boldsymbol{x} = 0, \quad \forall \Omega_0 \subseteq \Omega, \qquad \Rightarrow \quad \nabla \cdot [F\boldsymbol{v}] = 0, \quad \Omega. \tag{2.13}$$

Under the above assumptions the following is proved in the next section.

Theorem 1 *The operator A generates a semigroup $T(t) \in \mathcal{C}^0([0,\infty), \mathcal{L}(L^2(\Omega)))$. Also there exists an $\omega > 0$ such that*

$$\|T(t)\|_{L^2(\Omega)} \leq e^{-\omega t} \tag{2.14}$$

Further, for every $X_0 \in L^2(\Omega)$ and for every $U \in \mathcal{C}^{0,\gamma}([0,\infty), L^2(\Omega))$, $\gamma \in (0,1]$, X in (2.2) solves (2.1) and satisfies $X \in \mathcal{C}^0([0,\infty), L^2(\Omega)) \cap \mathcal{C}^1((0,\infty), L^2(\Omega))$ and $X(t) \in D(A)$ for every $t > 0$.

Given the solution X to (2.1), the associated solution C to (1.1) is given according to (2.2), (2.9) and (2.10):

$$C(t) - \bar{C}(t) = T(t)[C_0 - \bar{C}(0)] - \int_0^t T(t-s)\bar{C}'(s)ds \tag{2.15}$$

which is simplified formally with $T(t) = \exp[-At]$ by integrating by parts to obtain:

$$C(t) = T(t)C_0 + \int_0^t AT(t-s)I_\Omega C_{\text{AIF}}(s)ds \tag{2.16}$$

where (2.8) has been used to substitute $\bar{C}$. Note in (2.16) the elegant parallel with formula (2.4) in [2] for the case that Ω is well mixed leaving A to be a constant. For the more general case at hand, define:

$$C_{\text{T}}(t) = \frac{1}{|\Omega|} \int_\Omega C(\boldsymbol{x}, t) d\boldsymbol{x} \tag{2.17}$$

and

$$K(t) = \frac{1}{|\Omega|} \int_\Omega AT(t) I_\Omega d\boldsymbol{x} \tag{2.18}$$

to obtain (1.2) from (2.16) provided that

$$C_0 = 0. \tag{2.19}$$

Under the above assumptions the following is proved in the next section.

Theorem 2 *Given X in (2.2), the function $C = X + \bar{C}$ is a solution to (1.1) satisfying $C \in \mathcal{C}^0([0,\infty), L^2(\Omega)) \cap \mathcal{C}^1((0,\infty), L^2(\Omega))$ as well as (2.16). With vanishing initial conditions (2.19), the functions C_{T} in (2.17) and K in (2.18) satisfy (1.2).*

3 Proof of the Characterization

For completeness, the following theorems from [3], which are cited below, are stated here explicitly. Adapted for present purposes, Theorem 3.A, p. 89, from [3] follows.

Theorem 3 *Necessary and sufficient conditions that an operator $B : D(B) \subset L^2(\Omega) \to L^2(\Omega)$ be the generator of a semigroup $S(t) \in \mathcal{C}([0,\infty), \mathcal{L}(L^2(\Omega)))$ satisfying $\|S(t)\|_{\mathcal{L}(L^2(\Omega))} \le 1$, $\forall t \ge 0$, are that $D(B)$ be dense in $L^2(\Omega)$ and that the operator $(\lambda I - B) : D(B) \subset L^2(\Omega) \to L^2(\Omega)$ be a bijection satisfying $\|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \le 1$ for all $\lambda > 0$.* ∎

That these conditions are met for the current problem can be guaranteed according to the following adaptation of Theorem 6.A, p. 99, from [3].

Theorem 4 *Suppose that a in (2.4) is coercive:*

$$a(X, X) \ge \omega \|X\|^2_{H^1(\Omega)} \tag{3.1}$$

and bounded:

$$|a(X, Y)| \le \alpha \|X\|_{H^1(\Omega)} \|X\|_{H^1(\Omega)} \tag{3.2}$$

Then the operator A is uniquely defined by (2.5) and (2.6) and $D(A)$ is dense in $L^2(\Omega)$. Define $s(\theta) = \{z \in \boldsymbol{C} : |\arg(z)| < \theta\}$. Then $\exists \theta_0 \in (0, \frac{\pi}{4})$ such that $(\lambda I + A)^{-1} \in \mathcal{L}(L^2(\Omega))$, $\forall \lambda \in s(\frac{\pi}{2} + \theta_0)$. Also $\forall \theta \in (0, \theta_0)$, $\exists M_\theta$ such that $\|\lambda(\lambda I + A)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \le M_\theta$. In particular, $\forall \lambda > 0$ $\|\lambda(\lambda I + A)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \le 1$ holds. ∎

The conditions (3.1) and (3.2) are established below in Lemma 2. Since $(0, \infty) \subset s(\frac{\pi}{2} + \theta_0)$, $\forall \theta_0 \in (0, \frac{\pi}{4})$, the result of Theorem 4 is that the conditions of Theorem 3 are met. Hence, $-A$ generates a contraction semigroup $T(t) \in \mathcal{C}^0([0,\infty), \mathcal{L}(L^2(\Omega)))$ satisfying $\|T(t)\|_{\mathcal{L}(L^2(\Omega))} \le 1$, $\forall t \ge 0$. Nevertheless, it will be shown below that $\{T(t)\}_{t \ge 0}$ satisfies an exponential decay estimate. Furthermore, the next theorem shows that the semigroup is not just continuous but also analytic. This is an adaptation of Theorem 6.B, p. 100, from [3].

Theorem 5 *Let A be defined by (2.5) and (2.6). Let $s(\theta)$ and θ_0 be as given in Theorem 4. Then there is a family of operators $\{T(t) : t \in s(\theta_0)\} \subset \mathcal{L}(L^2(\Omega))$ satisfying the following:*

a. *$T(t+s) = T(s) \cdot T(s)$, $\forall t, s \in s(\theta_0)$ and $\forall X, Y \in L^2(\Omega)$, the function $t \mapsto \langle T(t)X, Y\rangle_{L^2(\Omega)}$ is analytic on $s(\theta_0)$.*

b. *For $t \in s(\theta_0)$, $T(t)$ is linear from $L^2(\Omega)$ into $D(A)$ and*

$$-\frac{dT}{dt}(t) = A \cdot T(t) \in \mathcal{L}(L^2(\Omega)) \tag{3.3}$$

c. *If $\theta \in (0, \theta_0)$, then for some constant $c(\theta)$,*

$$\|T(t)\|_{\mathcal{L}(L^2(\Omega))} \le c(\theta), \quad \|tAT(t)\|_{\mathcal{L}(L^2(\Omega))} \le c(\theta), \quad t \in s(\theta_0 - \theta) \tag{3.4}$$

and for $X \in L^2(\Omega)$, $T(t)X \to X$ as $t \to 0$, $t \in s(\theta_0 - \theta)$. ∎

Finally, the solution to (2.1) with the required regularity is given by the following adaptation of Theorem 6.E, p. 103, from [3].

Theorem 6 *Let A and $T(t)$ be given as in Theorem 4. Then for each $X_0 \in L^2(\Omega)$ and for each $U \in \mathcal{C}^{0,\gamma}([0,\infty), L^2(\Omega))$, X in (2.2) solves (2.1) satisfying $X \in \mathcal{C}^0([0,\infty), L^2(\Omega)) \cap \mathcal{C}^1((0,\infty), L^2(\Omega))$ and $X(t) \in D(A)$ for $t > 0$.* ∎

To establish conditions (3.2) and (3.1) of Theorem 4, the following technical lemma is used.

Lemma 1 *Assume that (2.13) holds and that $X \in H^1_\Gamma(\Omega)$. Then*

$$\int_\Omega X\nabla\cdot[F\boldsymbol{v}X]d\boldsymbol{x} = \frac{1}{2}\int_\Sigma X^2[F\boldsymbol{v}\cdot\boldsymbol{n}]d\boldsymbol{s}(\boldsymbol{x}) \geq 0 \tag{3.5}$$

Proof: According to Green's theorem [1],

$$\int_\Omega X\nabla\cdot[F\boldsymbol{v}X]d\boldsymbol{x} = \int_{\partial\Omega} X^2[F\boldsymbol{v}\cdot\boldsymbol{n}]d\boldsymbol{s}(\boldsymbol{x}) - \int_\Omega XF\boldsymbol{v}\cdot\nabla X d\boldsymbol{x}. \tag{3.6}$$

On the other hand, from the product rule,

$$\nabla\cdot[F\boldsymbol{v}X] = X\nabla\cdot(F\boldsymbol{v}) + F\boldsymbol{v}\cdot\nabla X = F\boldsymbol{v}\cdot\nabla X \tag{3.7}$$

where the first term, $X\nabla\cdot(F\boldsymbol{v}) = \nabla\cdot(F\boldsymbol{v}) = 0$, vanishes due to (2.13). Multiplying (3.7) by X, integrating over Ω and summing with (3.6) gives:

$$\begin{aligned} 2\int_\Omega X\nabla\cdot[F\boldsymbol{v}X]d\boldsymbol{x} &= \int_{\partial\Omega} X^2[F\boldsymbol{v}\cdot\boldsymbol{n}]d\boldsymbol{s}(\boldsymbol{x}) \\ &= \int_\Sigma X^2[F\boldsymbol{v}\cdot\boldsymbol{n}]d\boldsymbol{s}(\boldsymbol{x}) + \int_\Gamma X^2[F\boldsymbol{v}\cdot\boldsymbol{n}]d\boldsymbol{s}(\boldsymbol{x}) \\ &= \int_\Sigma X^2[F\boldsymbol{v}\cdot\boldsymbol{n}]d\boldsymbol{s}(\boldsymbol{x}) \end{aligned} \tag{3.8}$$

where the integral over Γ vanishes since $X = 0$ holds there. After noting that the integrand is necessarily non-negative on Σ, the result (3.5) is obtained. ∎

Lemma 1 justifies the definition of the energy norm:

$$\|X\|_A^2 = a(X,X) = \langle D(\boldsymbol{v})\nabla X, \nabla X\rangle_{L^2(\Omega)} + \frac{1}{2}\int_\Sigma X^2[F\boldsymbol{v}\cdot\boldsymbol{n}]d\boldsymbol{s}(\boldsymbol{x}) \tag{3.9}$$

which follows from (2.4) and (3.5). Next, the conditions of Theorem 4 are proved.

Lemma 2 *There exist constants $\alpha, \omega > 0$ such that the bilinear form a of (2.4) satisfies (3.2) and (3.1).*

Proof: Coercivity is established by contradiction. If no such constant ω exists, there exists a sequence $\{\tilde{X}_k\}$ such that $\|\tilde{X}_k\|_A \overset{k\to\infty}{\longrightarrow} 0$ while $\|\tilde{X}_k\|_{H^1(\Omega)} \geq \epsilon > 0$. Setting $X_k = \tilde{X}_k/\|\tilde{X}_k\|_{H^1(\Omega)}$ gives:

$$\|X_k\|_{H^1(\Omega)} = 1, \quad \forall k \tag{3.10}$$

and $\|X_k\|_A \leq \epsilon^{-1}\|\tilde{X}_k\|_A$ or:

$$\|X_k\|_A \longrightarrow 0, \quad k\to\infty. \tag{3.11}$$

Since the sequence $\{X_k\}$ is bounded in the Hilbert space $H^1_\Gamma(\Omega)$, there is a subsequence (again denoted by $\{X_k\}$) which converges weakly in $H^1_\Gamma(\Omega)$ to some $\hat{X} \in H^1_\Gamma(\Omega)$ [1]. Since $H^1_\Gamma(\Omega) \subset H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the sequence $\{X_k\}$ converges strongly in $L^2(\Omega)$ to $\hat{X}$ [1]. According to (2.12), (3.5) and (3.11),

$$\delta\|\nabla X_k\|^2_{L^2(\Omega)} \leq \langle D(\boldsymbol{v})\nabla X_k, \nabla X_k\rangle_{L^2(\Omega)} \leq \|X_k\|_A^2 \overset{k\to\infty}{\longrightarrow} 0 \tag{3.12}$$

the sequence $\{\nabla X_k\}$ converges to zero in $L^2(\Omega)$. Thus, the sequence $\{X_k\}$ actually converges strongly in $H^1_\Gamma(\Omega)$, and due to the uniqueness of limits, the strong limit is $\hat{X} \in H^1_\Gamma(\Omega)$. According to (3.12),

$$0 = \lim_{k\to\infty} \|\nabla X_k\|_{L^2(\Omega)} = \|\nabla \hat{X}\|_{L^2(\Omega)} \tag{3.13}$$

so $\hat{X}$ is a constant. Since $\hat{X} \in H^1_\Gamma(\Omega)$, $\hat{X} = \hat{X}|_\Gamma = 0$. However, this violates the following consequence of (3.10):

$$|\hat{X}||\Omega|^{\frac{1}{2}} = \|\hat{X}\|_{L^2(\Omega)} = \lim_{k\to\infty} \|\hat{X}\|_{L^2(\Omega)} = 1. \tag{3.14}$$

The contradiction implies coercivity of a.

Finally, from the estimate,

$$\begin{aligned} |a(X,Y)| &\leq (\|d\|_{L^\infty(\Omega)} + 2\|p\|_{L^\infty(\Omega)})\|\nabla X\|_{L^2(\Omega)}\|\nabla Y\|_{L^2(\Omega)} \\ &+ \|F\|_{L^\infty(\Omega)}\|\nabla X\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)} \\ &\leq (\|d\|_{L^\infty(\Omega)} + 2\|p\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(\Omega)})\|X\|_{H^1(\Omega)}\|Y\|_{H^1(\Omega)} \end{aligned} \tag{3.15}$$

boundedness of a follows. ∎

With Lemma 2 the results of Theorems 4, 5 and 6 follow immediately. The desired exponential decay estimate is established as follows.

Next, Lemma 1 and 2 are used to prove the following range condition for A.

Lemma 3 *The operator $B = \omega I - A$ satisfies the conditions of Theorem 3.*

Proof: Let $\lambda > 0$ be arbitrary and suppose $U \in L^2(\Omega)$. A weak solution to $(\lambda I - B)X = U$ is next constructed. For this, define the bilinear form on $H^1_\Gamma(\Omega) \times H^1_\Gamma(\Omega)$,

$$b(X,Y) = a(X,Y) + (\lambda - \omega)\langle X, Y\rangle_{L^2(\Omega)} \tag{3.16}$$

and the linear form on $H^1_\Gamma(\Omega)$,

$$u(Y) = \langle U, Y\rangle_{L^2(\Omega)} \tag{3.17}$$

The bilinear form is bounded according to (3.2):

$$\begin{aligned} |b(X,Y)| &\leq |a(X,Y)| + (\lambda+\omega)|\langle X,Y\rangle_{L^2(\Omega)}| \\ &\leq \alpha\|X\|_{H^1(\Omega)}\|Y\|_{H^1(\Omega)} + (\lambda+\omega)\|X\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)} \\ &\leq (\alpha+\lambda+\omega)\|X\|_{H^1(\Omega)}\|Y\|_{H^1(\Omega)} \end{aligned} \tag{3.18}$$

as is the linear form:

$$|u(Y)| \leq \|U\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)} \leq \|U\|_{L^2(\Omega)}\|Y\|_{H^1(\Omega)} \tag{3.19}$$

The coercivity of the bilinear form is established using (3.1),

$$\begin{aligned} b(X,X) &= a(X,X) + (\lambda-\omega)\langle X,X\rangle_{L^2(\Omega)} \\ &\geq \omega\|X\|^2_{H^1(\Omega)} + (\lambda-\omega)\|X\|^2_{L^2(\Omega)} \\ &= \omega\|\nabla X\|^2_{L^2(\Omega)} + \lambda\|X\|^2_{L^2(\Omega)} \\ &\geq \min\{\omega,\lambda\}\|X\|^2_{H^1(\Omega)} \end{aligned} \tag{3.20}$$

Thus, by the Lax-Milgram Theorem [1] there exists a unique $X \in H^1_\Gamma(\Omega)$ such that

$$b(X,Y) = u(Y), \quad \forall Y \in H^1_\Gamma(\Omega) \tag{3.21}$$

Using (3.16) and (2.5), B is related to b according to:

$$\begin{aligned} \langle -BX, Y\rangle_{L^2(\Omega)} &= \langle AX - \omega X, Y\rangle_{L^2(\Omega)} \\ &= a(X,Y) - \omega\langle X, Y\rangle_{L^2(\Omega)} \\ &= b(X,Y) + \lambda\langle X, Y\rangle_{L^2(\Omega)} \end{aligned} \tag{3.22}$$

Since the bilinear form on the right in (3.22) is bounded and coercive on $H^1_\Gamma(\Omega) \times H^1_\Gamma(\Omega)$, Theorem 4 implies that $B = \omega I - A$ and $D(B)$ are uniquely determined in analogy to (2.5) and (2.6). Since X satisfies:

$$\begin{aligned} |b(X,Y) + \lambda\langle X, Y\rangle_{L^2(\Omega)}| &= |u(Y) + \lambda\langle X, Y\rangle_{L^2(\Omega)}| \\ &\leq (\|U\|_{L^2(\Omega)} + \lambda\|X\|_{L^2(\Omega)})\|Y\|_{L^2(\Omega)}, \quad \forall Y \in L^2(\Omega) \end{aligned} \tag{3.23}$$

it follows that $X \in D(B)$. From (3.21) and (3.22) it follows that

$$\langle(\lambda I - B)X, Y\rangle_{L^2(\Omega)} = b(X,Y) = u(Y) = \langle U, Y\rangle_{L^2(\Omega)}, \quad \forall Y \in H^1_\Gamma(\Omega) \tag{3.24}$$

or $(\lambda I - B)X = U$. Thus, the operator $(\lambda I - B) : D(B) \subset L^2(\Omega) \to L^2(\Omega)$ be a bijection. To estimate the resolvent of B, note that the following is obtained by discarding $\omega\|\nabla X\|^2_{L^2(\Omega)} > 0$ from (3.20):

$$\begin{aligned} \lambda\|X\|^2_{L^2(\Omega)} &\leq b(X,X) = a(X,X) + (\lambda - \omega)\|X\|^2_{L^2(\Omega)} \\ &= \langle AX + (\lambda-\omega)X, X\rangle_{L^2(\Omega)} = \langle(\lambda I - B)X, X\rangle_{L^2(\Omega)} \\ &\leq \|(\lambda I - B)X\|_{L^2(\Omega)}\|X\|_{L^2(\Omega)} \end{aligned} \tag{3.25}$$

Therefore, $\|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \leq 1$, and the conditions of Theorem 3 are met. ■

From the calculation,

$$\begin{aligned} h^{-1}[e^{\omega h}T(h) - I]X &= h^{-1}[(e^{\omega h} - 1)T(h) + (T(h) - I)]X \\ &\overset{h\to 0}{\longrightarrow} (\omega I - A)X = BX, \quad X \in D(A) = D(B) \end{aligned} \tag{3.26}$$

it follows that the semigroup generated by $B = (\omega I - A)$, as guaranteed by Lemma 3 and Theorem 3, is $e^{\omega t}T(t)$ where $T(t)$ is the semigroup generated by A, as guaranteed by Theorems 3 and 4. Since B generates a contraction semigroup, $\|e^{\omega t}T(t)\|_{\mathcal{L}(L^2(\Omega))} \leq 1$ holds, and the decay estimate (2.14) follows. Thus, Theorem 1 follows with Theorem 6.

According to Theorem 1, the function C defined in (2.10) satisfies (2.15). The following is proved in order to justify the formal integration by parts leading to (2.16).

Lemma 4 *With $\bar{C}$ given in (2.8), the function C in (2.15) satisfies (2.16).*

Proof: In order to avoid the point $s = t$ in (2.15) at which the differentiability of $T(t)$ is not guaranteed by Theorem 5, (2.15) is divided as follows for $0 < t - \epsilon < t$:

$$C(t) - \bar{C}(t) = T(t)[C_0 - \bar{C}(0)] - \int_0^{t-\epsilon} T(t-s)\bar{C}'(s)ds - \int_{t-\epsilon}^t T(t-s)\bar{C}'(s)ds \tag{3.27}$$

Note that the integral in (3.27) over $[0, t-\epsilon]$ converges to the full integral in (2.15) as $\epsilon \to 0$ since the integral in (3.27) over $[t-\epsilon, t]$ vanishes according to:

$$\begin{aligned} \left\|\int_{t-\epsilon}^t T(t-s)\bar{C}'(s)ds\right\|_{L^2(\Omega)} &\leq \int_{t-\epsilon}^t \|T(t-s)I_\Omega\|_{L^2(\Omega)}|C'_{\text{AIF}}(s)|ds \\ &\leq \sup_{0\leq s\leq t}|C'_{\text{AIF}}(s)|\int_{t-\epsilon}^t e^{-\omega(t-s)}\sqrt{|\Omega|}ds \overset{\epsilon\to 0}{\longrightarrow} 0 \end{aligned} \tag{3.28}$$

Then the integral in (3.27) over $[0, t-\epsilon]$ is simplified by noting in Theorem 5b that $(0, \infty) \subset s(\theta_0)$:

$$\begin{aligned}\int_0^{t-\epsilon} T(t-s)\bar{C}'(s)ds &= [T(t-s)\bar{C}(s)]_0^{t-\epsilon} - \int_0^{t-\epsilon} \frac{d}{ds}T(t-s)\bar{C}(s)ds \\ &= [T(\epsilon)\bar{C}(t-\epsilon) - T(t)\bar{C}(0)] - \int_0^{t-\epsilon} AT(t-s)\bar{C}(s)ds\end{aligned} \tag{3.29}$$

The last integral in (3.29) converges in $L^2(\Omega)$ as $\epsilon \to 0$ since all other terms in (3.29) converge in $L^2(\Omega)$. Thus, combining (3.27) – (3.29) gives:

$$C(t) - \bar{C}(t) = T(t)[C_0 - \bar{C}(0)] - [T(0)\bar{C}(t) - T(t)\bar{C}(0)] + \int_0^t AT(t-s)\bar{C}(s)ds \tag{3.30}$$

which simplifies to (2.16). ∎

Finally, average values in (2.16) are taken over Ω and Fubini's Theorem [1] is used to reverse temporal and spatial integrations to obtain the formulas (2.17) and (2.18). With vanishing initial conditions in (2.19), the desired convolution formula (1.2) is obtained. Thus, Theorem 2 is proved.

References

[1] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, 1998.

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[3] R.E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, San Francisco, 1979.