Update on singularity categories

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based on arXiv: 2108.03292

(cf. also 2103.06584)

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Motivation

(Representation theory)

R (noetherian) ring.

<u>Goal</u>: Understand / Classify all R-modules.

<u>Well-known problem</u>:

This is almost always "hopeless"!

$\rightsquigarrow$ <u>Today,</u> restrict to "understanding" a subcategory of R-$\underline{\text{mod}}$:

$$\Omega^{\infty}(R) := \left\{ M \in R\text{-}\underline{\text{mod}} \;\middle|\; \begin{array}{l} \text{for all } n \geq 0, \text{ there is} \\ N \in R\text{-}\underline{\text{mod}}, \text{ s. th.} \\ M \in \underline{\text{add}}\text{-}\Omega^{n}(N) \end{array} \right\}$$

Think of $\underline{\Omega}^{\infty}(R)$ as "stable range"/"stabilization" of syzygy functor $\underline{\Omega}$.

<u>Examples:</u>

(a) $\operatorname{gldim} R < \infty \Rightarrow \underline{\Omega}^{\infty}(R) = 0$.

(b) $\underline{\Omega}^{\infty}\left(\dfrac{\mathbb{C}[x,y,z]}{(x,y,z)^2}\right) \simeq \underline{\text{add}}\text{-}S$, $\quad S \simeq \dfrac{\mathbb{C}[x,y,z]}{(x,y,z)}$ <u>simple</u>

$\cong \mathbb{C}\left(\begin{tikzcd} \bullet \arrow[loop above] \arrow[loop left] \arrow[loop right] \end{tikzcd}\right) / (\text{arrow ideal})^2$ ("wild")

(b') $\underline{\Omega}^{\infty}\left(\mathbb{C}\left(\begin{tikzcd} 1 \arrow[loop above] \arrow[loop left] \arrow[loop right] & 2 \arrow[l] \end{tikzcd}\right) / (\text{arrow ideal})^2\right) \simeq \underline{\text{add}}\text{-}S_1$

(c) R selfinjective $\Rightarrow \underline{\Omega}^{\infty}(R) \simeq R\text{-}\underline{\text{mod}}$.

(d) R Gorenstein, i.e. $\text{inj.dim}_R R, R_R < \infty$

$\Rightarrow \underline{\Omega}^{\infty}(R) \simeq \underline{\text{GP}}(R) := \left\{ M \in R\text{-mod} \;\middle|\; \text{Ext}_R^{>0}(M,R) = 0 \right\}$

In particular, in (c) & (d) $\underline{\Omega}^{\infty}(R)$ admits triang. struct.

A finite representation type classification for $\Omega^\infty(R)$, where R complete local Gorenstein $\mathbb{C}$-algebra.

Thm [Auslander-Reiten, Eisenbud, Knörrer, Herzog
Buchweitz-Greuel-Schreyer, ...]

$R = \mathbb{C}[[z_0, \dots, z_d]]/I$ <u>Gorenstein</u>.

If $\Omega^\infty(R) \neq 0$ has finite repr. type

<u>THEN</u> $R \cong \mathbb{C}[[z_0, \dots, z_d]]/(f) =: P_d/(f)$

is an ADE-hypersurface.

[e.g. $f = z_0^{n+1} + z_1^2 + \dots + z_d^2$ (A_n)]

Moreover, in this case, TFAE:

(1) $\Omega^\infty(P_d/(f)) \cong \Omega^\infty(P_e/(g))$ as add. cats

(2) $\Omega^\infty(P_d/(f)) \cong \Omega^\infty(P_e/(g))$ as Δ-cats

(3) $|d-e| = 2n$ <u>and</u> $f - g = z_1^2 + \dots + z_{2n}^2$

(after suitable "coordinate change")

<u>Rem:</u>

Today we will generalize the last part of Thm. to all isolated hypersurface singularities!

Matrix factorizations

Let $0 \neq f \in \mathbb{C}[\![z_0, \dots, z_d]\!] =: S$

A <u>matrix factorization</u> (MF) of f is a pair

$$(A, B) \in \mathrm{Mat}_n(S) \times \mathrm{Mat}_n(S)$$

satisfying

$$A \cdot B = f \cdot \mathrm{Id}_n = B \cdot A$$

Examples:

$(\mathrm{Id}_n, f\cdot \mathrm{Id}_n)$ and $(f\cdot \mathrm{Id}_n, \mathrm{Id}_n)$

are trivial matrix factorizations

Example: $f = z_0^2 + z_1^2 + z_2^2 + z_3^2$

Step 1: Consider $g = z_0^2 + z_1^2$

$$g = (z_0 + iz_1)(z_0 - iz_1) =: XY \quad \text{is MF}$$

Step 2:

$$\begin{pmatrix} -X & (z_2 - iz_3) \\ (z_2 + iz_3) & Y \end{pmatrix}, \begin{pmatrix} -Y & (z_2 - iz_3) \\ (z_2 + iz_3) & X \end{pmatrix}$$

is MF for f.

Remark:

(a) A similar approach yields a Matrix fact.

$$A \cdot A = (z_0^2 + z_1^2 + z_2^2 + z_3^2) \cdot Id_4$$

which is used in Dirac's dicovery of the Dirac equation.

(b) More recently, theoretical physicists are interested in understanding and classifying MFs in relation with 2D-Quantum Field Theories.

Example: $f = z_0^2 + z_1^2 + z_2^2 + z_3^2$

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$$\begin{pmatrix} -X & (z_2 - iz_3) \\ (z_2 + iz_3) & Y \end{pmatrix}, \begin{pmatrix} -Y & (z_2 - iz_3) \\ (z_2 + iz_3) & X \end{pmatrix}$$

is MF for f.

<u>Observation</u> (Knörrer 1987):

Let $0 \neq g \in \mathbb{C}[[z_0, \dots, z_d]]$

The construction in Step 2 defines a map

$$MF(g) \longrightarrow MF(g + z_{d+1}^2 + z_{d+2}^2).$$

This is a <u>bijection</u> (up to sums of trivial MFs).

More precisely, MFs (A,B) of f yield 2-periodic diagrams

$$\cdots \xrightarrow{A} S^n \xrightarrow{B} S^n \xrightarrow{A} S^n \xrightarrow{B} \cdots$$

which (analogous to chain complexes) form a differential $\mathbb{Z}$-graded category

$$MF(f)$$

with degree d maps $m \in \mathrm{Hom}^d_{MF(f)}\big((A,B),(A',B')\big)$

(e.g. for $d=-2$: $m=(m_i)_i$

$$\begin{tikzcd}
\cdots \arrow[r, "A"] & S^n \arrow[r, "B"] \arrow[dl, "m_i"'] & S^n \arrow[r, "A"] \arrow[dl, "m_{i+1}"'] & S^n \arrow[r, "B"] \arrow[dl, "m_{i+2}"'] & S^n \arrow[r, "A"] \arrow[dl, "m_{i+3}"'] & S^n \arrow[r] \arrow[dl, "m_{i+4}"'] & \cdots \\
\cdots \arrow[r, "A'"] & S^m \arrow[r, "B'"] & S^m \arrow[r, "A'"] & S^m \arrow[r, "B'"] & S^m \arrow[r, "A'"] & S^m \arrow[r] & \cdots
\end{tikzcd}$$

and differential

$$d(m) = m\, d_{(A,B)} - (-1)^{|m|} d_{(A',B')}\, m$$

(where $d^i_{(A,B)} = \begin{cases} A & i \text{ even} \\ B & i \text{ odd} \end{cases}$, similar for $d_{(A',B')}$)

In analogy with homotopy categories of complexes one can define the <u>homotopy category of MFs</u> $[MF(f)]$ with morphism spaces

$$H^0\left(\mathrm{Hom}^\bullet_{MF(f)}\left((A,B),(A',B')\right)\right).$$

<u>Thm</u> (<u>Knörrer '87</u>) There is a triangle equiv.

$$[MF(f)] \xrightarrow{\sim} [MF(f + z_{d+1}^2 + z_{d+2}^2)]$$

The Buchweitz – Orlov Singularity Category

X quasi-proj. variety $/\mathbb{C}$

$$\mathrm{Perf}(X) \hookrightarrow D^b(\mathrm{Coh}\, X) \twoheadrightarrow D_{sg}(X) := \frac{D^b(\mathrm{Coh}\, X)}{\mathrm{Perf}(X)}$$

'smooth part' consisting of bounded complexes of vector bundles

Singularity Category measures complexity of singularities of X

Thm (Auslander-Buchsbaum & Serre)

$$X \text{ smooth} \iff D_{sg}(X) = 0$$

<u>Thm</u> (Orlov) If X has isolated singularities

$$D_{sg}(X) \cong \bigoplus_{s \in \mathrm{Sing}(X)} D_{sg}(\hat{\mathcal{O}}_s)$$

(up to taking direct summands)

where

$$D_{sg}(R) := \frac{D^b(\text{mod-}R)}{K^b(\text{proj-}R)}$$

<u>singularity category</u> of a noetherian ring R.

Thm (Buchweitz, Eisenbud)

Let $0 \neq f \in \mathbb{C}[[z_0, \dots, z_d]] =: S$

$$\underline{\Omega}^{\infty}(S/(f)) \cong [MF(f)] \cong D_{sg}(S/(f))$$

Def: Two $\begin{pmatrix}\text{possibly} \\ \text{noncomm.}\end{pmatrix}$ algebras R, S are called singular equivalent if there is a triangle equivalence

$$D_{sg}(R) \cong D_{sg}(S)$$

Complete list of known singular equivalences between commutative complete local $\mathbb{C}$-algebras:

(0) $D_{sg}(R) = 0 = D_{sg}(S)$, if $\mathrm{gldim}\, R, S < \infty$

(1) $D_{sg}\left(\frac{P_d}{(f)}\right) \cong D_{sg}\left(\frac{P_d[[y_1, \ldots, y_{2n}]]}{(f + y_1^2 + \ldots + y_{2n}^2)}\right)$, for $f \neq 0$

[Knörrer '87]

Let $\mathbb{C}[[z_1, \ldots, z_d]]^{\frac{1}{m}(a_1, \ldots, a_d)}$ be the invariant ring of the following group action:

$$z_i \longmapsto \varepsilon_m^{a_i} z_i \quad \left(\begin{array}{c}\text{for a primitive} \\ m\text{-th root of unity} \\ \varepsilon_m \in \mathbb{C}\end{array}\right)$$

(2) $D_{Sg}\left(\mathbb{C}[[z_1,z_2]]^{\frac{1}{m}(1,1)}\right) \cong D_{Sg}\left(\dfrac{\mathbb{C}[[z_1,\dots,z_{m-1}]]}{(z_1,\dots,z_{m-1})^2}\right)$

[D. Yang, Y. Kawamata, K.–Karmazyn]
all ~ 2015

(3) $D_{Sg}\left(\mathbb{C}[[z_1,z_2,z_3]]^{\frac{1}{2}(1,1,1)}\right) \cong D_{Sg}\left(\mathbb{C}[[z_1,z_2]]^{\frac{1}{4}(1,1)}\right)$

[K. 2021]

<u>Observation:</u>

Knörrer's equivalences are the only known non-trivial Gorenstein examples.

Goal:

Explain this observation (to some extent), i.e. explain why we (maybe) shouldn't expect to find (many) more equivalences.

1. Step:

Reduction to hypersurfaces

Thm [cf. K.'21 and also Puthenpurakal '21]

$R_j = P_{d_j}/I_j$ complete local $\mathbb{C}$-algebras, s.th

$$\Phi : D_{sg}(R_1) \cong D_{sg}(R_2) \text{ as } \Delta\text{-ted cats}$$

and R_1 is Gorenstein with isolated sing.

If $\begin{cases} R_1 = P_{d_1}/(f_1) \text{ hypersurface } \underline{\text{OR}} \\ \operatorname{krdim} R_1 \neq \operatorname{krdim} R_2 \end{cases}$

Then both $R_j \cong P_{d_j}/(f_j)$ are hypersurfaces

(with isolated singularities)

Sketch of proof:

Step 1: R_1 Gorenstein isolated singularity

$\overset{\text{Auslander}}{\Longrightarrow}$ $D_{sg}(R_1)$ Hom-finite

$\Longrightarrow$ $D_{sg}(R_2) \cong D_{sg}(R_1)$ Hom-finite

cf. Avramov & Veliche $\Longrightarrow$ R_2 Gorenstein isolated singularity

Step 2: $\exists\, 0 \neq n \in \mathbb{Z}$, s.th.

$$[n] \cong \mathrm{id} \quad \text{in} \quad D_{sg}(R_i)$$

Indeed, if R_1 hypersurface $\Longrightarrow$ $[2] \cong \mathrm{id}$

if $d_1 := \mathrm{krdim}\, R_1 \neq \mathrm{krdim}\, R_2 =: d_2$

Auslander, Bondal & Kapranov

$$\Longrightarrow [d_1 - 1] \cong S_{D_{sg}(R_1)} \cong \Phi^{-1} S_{D_{sg}(R_2)} \Phi \cong [d_2 - 1]$$

$$\Longrightarrow [d_1 - d_2] \cong \mathrm{id}$$

<u>Step 3:</u> $[n] \cong \mathrm{id} \Longrightarrow$ the R_i-modules

$R_i/m_i \cong \mathbb{C}$ have bounded Betti-numbers

$\overset{\text{Gulliksen}}{\Longrightarrow}$ $R_i \cong P_{d_i}/(f_i)$ are hypersurfaces

as claimed.

2. Step :

Classifying dg singularity categories of hypersurfaces up to quasi-equivalence

DG-singularity categories

For hypersurface singularities g have seen dg cat. $MF(g)$ s.th. $[MF(g)] \cong D_{sg}(P_d/(g))$.

More generally, can define $D_{sg}^{dg}(R)$ as dg quotient [cf. Keller, Drinfeld]

$$D^b_{dg}(R\text{-mod}) / \mathrm{Perf}_{dg}(R)$$

and show

$$MF(g) \underset{\text{quasi-eq.}}{\cong} D_{sg}^{dg}(P_d/(g))$$

[Blanc-Robalo-Toën-Vezzosi]

The following
result answers
a question of
Keller & Shinder.

Thm [K. '21]

Let $0 \neq f_j \in P_{d_j}$, $d_1 \geq d_2$ and assume that f_1 has <u>isolated</u> singularity.

TFAE

(a) $D^{dg}_{sg}\left(P_{d_1}/(f_1)\right) \cong D^{dg}_{sg}\left(P_{d_2}/(f_2)\right)$ $\mathbb{C}$-linear quasi-equiv. of dg-cats.

(b) $P_{d_1}/(f_1) \cong P_{d_1}/(g_1)$, s.th.

$g_1 - f_2 = z^2_{d_2+1} + \dots + z^2_{d_1}$ <u>and</u> $d_1 - d_2 = 2n$.

<u>Pf (Sketch)</u> <u>(b) $\Rightarrow$ (a) [Knörrer; dg version: Dyckerhoff and also Orlov]</u>

$T(f) := P_d/(f, \partial_0 f, \dots, \partial_d f)$ <u>Tyurina algebra</u>

(a) $\xRightarrow{\text{[Hua-Keller]}}$ $T(f_1) \cong T(f_2)$

⌜Using Hochschild cohomology of dg categories, which is equiv. to Buchweitz's singular Hochschild cohom.

by work of Keller inspired by Zhengfang Wang's thesis. $\perp$

<u>Fact</u>: $T(f) \cong T(f + y_1^2 + \ldots + y_t^2)$

In combination with discussion above, we get

$$T(f_1) \cong T(f_2) \cong T(\underbrace{f_2 + z_{d_2+1}^2 + \ldots + z_{d_1}^2}_{=: g_1 \in P_{d_1}})$$

<u>Thm</u> (Mather-Yau, cf. also Greuel-Pham)

$h_i \in P_d$, s.th. $T(h_1) \cong T(h_2)$ as algebras
(isol. sing.)

$$\Longrightarrow P_d/(h_1) \cong P_d/(h_2)$$

Applying this to $f_1, g_1 \in P_{d_1}$ yields

$$P_{d_1}/(f_1) \cong P_{d_1}/(f_2 + z_{d_2+1}^2 + \ldots + z_{d_1}^2)$$

Using Knörrer's work one can check that

$$D_{sg}\left(P_d/(h)\right) \not\simeq D_{sg}\left(P_{d+1}/(h+z_{d+1}^2)\right)$$

implying $d_1 - d_2$ is even and completing the proof.

<u>Rem:</u>

(a) Let $p \in \mathbb{Z}$ be prime. There are Δ-equiv.

$$D_{sg}\left(\mathbb{Z}/p^2\mathbb{Z}\right) \simeq \mathbb{Z}/p^2\mathbb{Z}\text{-}\underline{\mathrm{mod}} \simeq \mathbb{Z}/p\text{-mod} \simeq \frac{\mathbb{Z}/p[x]}{(x^2)}\text{-}\underline{\mathrm{mod}} \simeq D_{sg}\left(\frac{\mathbb{Z}/p[x]}{(x^2)}\right)$$

However, these categories have dg-enhancements which are <u>not</u> quasi-equivalent if $p \neq 2$ (Schlichting '02)

(b) This does <u>not</u> happen for k-linear enhancements of connected representation finite categories over perfect fields k! [Muro '18, cf. also Keller '18 and K.-Yang '16]

More generally, k-linear dg-enhancements of triangulated categories admitting a <u>n-cluster tilting subcategory</u> $\mathcal{C}$ s.th.

$$\mathcal{C}[n] \cong \mathcal{C}$$

are <u>unique</u>! [Jasso & Muro '22]

. .

<u>Corollary</u>_ Let R_i be complete local comm. $\mathbb{C}$-algebras.

Let R_1 be an

(a) ADE-hypersurface singularity <u>or</u>

(b) 3 dim^l, isolated cDV singularity admitting a small resolution of singularities.

The following statements are equivalent:

(i) $D_{sg}(R_1) \cong D_{sg}(R_2)$ as Δ-ted cats

(ii) $D^{dg}_{sg}(R_1) \cong D^{dg}_{sg}(R_2)$ quasi-equiv. of dg-cats

(iii) There exist isom. $R_i = P_{d_i}/(f_i)$, s. th.

$f_1 - f_2 = z^2_{d_2+1} + \dots + z^2_{d_1}$ <u>and</u> $d_1 - d_2 = 2n$.

(where w.l.o.g $d_1 \geq d_2$)

In particular, we get

a generalization of the classification of equivalences between the cats $\Omega^\infty(R)$ stated in the beginning.

The End.

Thank you very much!

if you have comments or questions later, you are very welcome to send me an email:

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