Semiorthogonal decompositions on singular varieties

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(joint work with N. Pavic & E. Shinder)

Categories and birational geometry

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1. Introduction & Motivation

Setup:

X Gorenstein projective variety $/_{\mathbb{C}}$

Aim: Understand structure of
$$D^b(X) := D^b(\mathrm{coh}\, X)$$
using semiorthogonal decompositions (SOD)

Examples of SODs related to <u>rationality questions</u>:

$$D^b(X) = \langle E_1, \dots, E_n, \mathcal{A} \rangle \qquad (*)$$

with E_i exceptional and

$$\mathcal{A} \subset \mathrm{Perf}(X)$$

<u>In particular</u>: if $\mathcal{A} = 0 \rightsquigarrow$ full exc. seq.

<u>Observation</u>:

X singular $\Rightarrow D^b(X)$ has $\underline{\underline{\text{no}}}$ SODs of the form $(*)$

In conjunction with recent work of Kawamata, this motivates

Def.: An admissible SOD

$D^b(X) = \langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m \rangle$ with $m \geq 1$ and

$$\mathcal{C}_i \cong \begin{cases} D^b(R_i),\ R_i \text{ finite dim } \mathbb{C}\text{-algebra} & \text{OR} \\ \mathcal{A}_i \subseteq \mathrm{Perf}(X) \end{cases}$$

is called Kawamata semiorthogonal dec. (KSOD)

Rem: (a) The R_i are allowed to be non-comm.

They "contain the information about singularities of X."

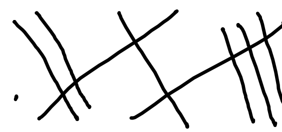
(b) for $m \leq 2$ admissibility always holds.

Examples

a) X smooth $\Rightarrow$ all SODs are KSODs

b) If X has a <u>tilting obj.</u> $T \in D^b(X)$ then

$D^b(X) \cong D^b(\mathrm{End}_X(T))$ is a KSOD.

<u>Ex (Burban)</u>: $X =$ tree of $\mathbb{P}^1$'s, e.g. 

$D^b(\mathbb{P}^1 \vee \mathbb{P}^1 = X) = \langle \mathcal{O}, D^b(R_{nd}) \rangle$ with

$R_{nd} = \begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "b"] \end{tikzcd}$ with relations $ab = 0 = ba$

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c) X toric surface (not necessarily Gorenstein)

$D^b(X, \beta) \cong \langle D^b(R_1), \dots, D^b(R_m) \rangle$ SOD

(Karmazyn, Kuznetsov & Shinder '18)

with <u>explicit</u> finite dimensional local algebras R_i discovered in (K.-Karmazyn '17)

and $\beta \in \mathrm{Br}(X)$.

If X Gorenstein $\Longrightarrow$ SOD is admissible

and $R_i \cong \mathbb{C}[x]/(x^{n_i})$ with $n_i \geq 1$.

If <u>additionally</u> $Br(X)=0$ get KSOD

$$D^b(X) = \langle D^b(R_1), \dots, D^b(R_m) \rangle.$$

E.g. $X = \mathbb{P}(1,1,2)$ or $\mathbb{P}(1,2,3)$

where such decomp. were studied earlier by Kuznetsov and Kawamata.

d) $X \subset \mathbb{P}^4$ nodal quadric 3-fold

$$D^b(X) = \langle \mathcal{O}_X(-2H), \mathcal{O}_X(-H), D^b(R_{nd}), \mathcal{O}_X \rangle$$

(Kawamata and Kuznetsov)

e) X nodal del Pezzo 3-fold of degree 6

$$D^b(X) = \langle E_1, \dots, E_5, D^b(R_{nd}) \rangle$$

(Kawamata)

We can use these examples together with Orlov's "blow-up formula" to construct further varieties with KSODs:

Prop (KPS)

Let $\tilde{X} = Bl_Z(X)$, where $Z \subset X$ is a closed subvariety, which is a locally complete intersection.

If X and Z have KSODs, then

$\tilde{X}$ has a KSOD.

"Global" obstructions to Kawamata semiorthogonal decompositions

"Warm-up":

Lemma (KPS)

If X is a proj. Gorenstein variety with $\omega_X \cong \mathcal{O}_X$, $\dim X \geq 1$ and KSOD, then X is smooth.

idempotent complete

$$\underline{CM}(R_1) \quad \cdots \quad \underline{CM}(R_m)$$

$$\cong \qquad\qquad \cong \text{ [Buchweitz]}$$

$$\frac{D^b(X)}{\mathrm{Perf}(X)} =: D_{sg}(X) \cong \langle D_{sg}(R_1), \ldots, D_{sg}(R_m) \rangle$$

$$D_{sg}(R_1) \cong D^b(R_1) / \mathrm{Per}(R_1)$$

$$\begin{tikzcd}
D_{sg}(X) \\
D^b(X) \arrow[u, two heads] \\
\mathrm{Perf}(X) \arrow[u, hook]
\end{tikzcd}$$

after mutation

$$D^b(X) = \langle \mathcal{A}, D^b(R_1), \ldots, D^b(R_m) \rangle \qquad \text{(KSOD)}$$

using admissibility

$$\mathrm{Perf}(X) = \langle \mathcal{A}, \mathrm{Perf}(R_1), \ldots, \mathrm{Perf}(R_m) \rangle$$

$- \otimes \omega_X[\dim X]$ induces Serre functors

Serre functor (since X Gorenstein)

[Happel] R_i are Gorenstein, i.e.

$$\operatorname{injdim}_{R_i} R_i < \infty$$

$$\operatorname{injdim} {R_i}_{R_i} < \infty$$

Corollary (KPS)

If X has a KSOD then the following equivalent statements hold:

(a) $\mathcal{D}_{sg}(X)$ is idempotent complete.

(b) $K_{-1}(X) = 0$.

Ex: [Weibel, Burban] X connected proj. curve with only nodal singularities and all irred components are rational. TFAE

(i) X has a KSOD

(ii) $K_{-1}(X) = 0$

(iii) X is a tree of $\mathbb{P}^1$'s

Ex [Karmazyn, Kuznetsov & Shinder, KPS]

X proj. Gor. toric surface. TFAE

(i) X has a KSOD

(ii) $K_{-1}(X) = 0$

<u>Prop</u> (KPS) Let X normal irreducible 3-fold with isolated singularities. If $s \in \mathrm{Sing}(X)$ is a nodal singularity s.th.

$$\mathcal{O}_{X,s} \text{ is factorial}$$

then X has <u>no</u> KSOD.

<u>Rmk</u>:

This explains why Kawamata's construction of KSODs fails in such examples.

"Complete local"

obstructions to

Kawamata SODs

Observation:

All **known** odd-dimensional varieties with isolated singularities and KSOD are nodal.

Aim: Understand this observation to some extent. using singularity categories.

Setup:

X 3-dimensional (for simplicity, we remark on general case later)

proj. Gor. with isolated singularities.

Notation:

$\mathrm{ADE}(X) := \{ s \in \mathrm{Sing}(X) \mid \hat{\mathcal{O}}_{X,s} \text{ is an ADE-singularity} \}$

$\mathrm{NCCR}(X) := \{ s \in \mathrm{Sing}(X) \mid \hat{\mathcal{O}}_{X,s} \text{ is normal domain with NCCR} \}$

$\mathrm{SMALL}(X) := \{ s \in \mathrm{Sing}(X) \mid \mathrm{Spec}(\hat{\mathcal{O}}_{X,s}) \text{ has a small resolution} \}$

<u>Rem</u>: (i) $\mathrm{SMALL}(X) \subseteq \mathrm{NCCR}(X)$ by Van den Bergh

(ii) $\mathrm{SMALL}(X) \cap \mathrm{ADE}(X)$ can be non-empty.

Theorem (KPS). Assume Setup above.
If X has a KSOD and

$$\mathrm{Sing}(X) = \mathrm{ADE}(X) \cup \mathrm{NCCR}(X)$$

then the following implications hold

(a) $s \in \mathrm{SMALL}(X)$ or $s \in \mathrm{ADE}(X) \setminus \{A_{2n}\}$

$$\Longrightarrow \widehat{\mathcal{O}}_{X,s} \cong \frac{\mathbb{C}[[w,x,y,z]]}{(wx-yz)} \text{ is nodal}$$

(b) $\mathrm{Sing}(X) = \mathrm{ADE}(X) \cup \mathrm{SMALL}(X)$

$\Rightarrow$ X is nodal, i.e.

$$\widehat{\mathcal{O}}_{X,s} \cong \frac{\mathbb{C}[[w,x,y,z]]}{(wx-yz)} \quad \text{for all } s \in \mathrm{Sing}(X)$$

Rem:

Proof only uses the structure of the triangulated category

$$\mathcal{D}_{sg}(X) := D^b(X) / \mathrm{Perf}(X)$$

Knörrer's periodicity shows

$$D_{sg}\left(\frac{\mathbb{C}[x_0,\dots,x_d]}{(f)}\right) \cong D_{sg}\left(\frac{\mathbb{C}[x_0,\dots,x_d,y,z]}{(f+yz)}\right), \quad (\text{for } f \neq 0)$$

Thus there are odd-dimensional varieties Y with $\dim(Y) \neq 3$ and

$$\mathcal{D}_{sg}(Y) \cong \mathcal{D}_{sg}(X)$$

If X has no KSOD then Y has no KSOD

Idea of the proof

Recall from before ...

idempotent complete

$$
\begin{array}{ccc}
& \underline{CM}(R_1) \quad \cdots & \underline{CM}(R_m) \\
& \wr\!\!\parallel & \wr\!\!\parallel \text{ [Buchweitz]} \\
\dfrac{D^b(X)}{\mathrm{Perf}(X)} =: D_{sg}(X) \cong & \langle D_{sg}(R_1), & \ldots, D_{sg}(R_m) \rangle
\end{array}
$$

$$
\begin{tikzcd}
D_{sg}(X) \\
D^b(X) = \langle \mathcal{A}, D^b(R_1), \ldots, D^b(R_m) \rangle \arrow[u, two heads] \\
\mathrm{Perf}(X) = \langle \mathcal{A}, \mathrm{Perf}(R_1), \ldots, \mathrm{Perf}(R_m) \rangle \arrow[u, hook]
\end{tikzcd}
$$

(KSOD)

$- \otimes \omega_X [\dim X]$

Serre functor
(since X Gorenstein)

induces

Serre functors

[Happel] R_i are Gorenstein, i.e.

$\mathrm{injdim}_{R_i} R_i < \infty$

$\mathrm{injdim}\, {R_i}_{R_i} < \infty$

By Orlov and idempotent completeness of $D_{sg}(X)$:

$$D_{sg}(X) \cong \bigoplus_{s \in \mathrm{Sing}(X)} D_{sg}(\hat{\mathcal{O}}_{X,s})$$

By Auslander (+ Gor. assumption) these categories are 2-Calabi-Yau.

Thus

$$\begin{aligned} D_{sg}(X) &\cong \langle D_{sg}(R_1), \dots, D_{sg}(R_n) \rangle \\ &\cong D_{sg}(R_1) \oplus \dots \oplus D_{sg}(R_n) \\ &\cong D_{sg}(R) \end{aligned}$$

where $R = R_1 \times \dots \times R_n$ is finite dim. and Gorenstein

Assume for simplicity: $\mathrm{Sing}(X) = \mathrm{SMALL}(X)$

Van den Bergh $\Longrightarrow$ all $\hat{\mathcal{O}}_{X,s}$ have an NCCR

$\overset{\text{Iyama}}{\Longrightarrow}$ all $D_{sg}(\hat{\mathcal{O}}_{X,s})$ contain

<u>cluster-tilting objects</u> T_s

⌜In our situation, <u>cluster-tilting obj.</u> are precisely the <u>maximal rigid objects</u> M:

i.e. $\quad \mathrm{Hom}_{D_{sg}}(M, M[1]) = 0$ and if

$$\mathrm{Hom}_{D_{sg}}(M \oplus N, (M \oplus N)[1]) = 0 \text{ for some}$$

$N \in D_{sg}(\hat{\mathcal{O}}_{X,s})$ then $\quad N \oplus N' \cong M^{\oplus n}$⌟

$\Longrightarrow \bigoplus_{s \in \mathrm{Sing}(X)} T_s$ is cluster-tilting in $D_{sg}(X)$

Now $D_{sg}(X) \cong D_{sg}(R)$

are Hom-finite, 2-CY triangulated categories with cluster-tilting objects T.

In particular,

finite quiver

$$\mathrm{End}_{D_{sg}}(T) \cong \mathbb{C}Q/I$$

adm. ideal

is a finite dimensional algebra.

For $T \in D_{sg}(R) \cong \underline{CM}(R)$

cluster theory (cf. Buan-Iyama-Reiten-Scott)

shows that Q does <u>not</u> have

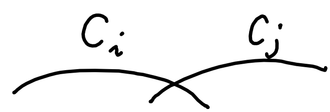
(1) $\circlearrowleft\bullet$ loops and

(2) $\bullet \rightleftarrows \bullet$ 2-cycles

For $T \in D_{sg}(X) \cong \bigoplus_{s \in Sing(X)} D_{sg}(\hat{\mathcal{O}}_{X,s})$

(cf. Wemyss's 3dim-Auslander-McKay corresp.)

$$\left\{\begin{array}{c}\text{Vertices in}\\ Q\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{l}\text{exceptional curves } C_i\\ \text{in small res. } Y_s \longrightarrow \operatorname{Spec} \hat{\mathcal{O}}_s\end{array}\right\}$$

$i \rightleftarrows j$ if 

no loops at vertex i iff C_i is $(-1,-1)$-curve

Comparing these statements shows that the exceptional fibre of all small resolutions

$$Y_s \longrightarrow \mathrm{Spec}(\hat{\mathcal{O}}_{X,s})$$

contains a single $(-1,-1)$-curve C. This implies that $\hat{\mathcal{O}}_{X,s}$ is an A_1-singularity and completes the proof.

Summary

* Kawamata SODs generalize the concept of full exc. seq. to the singular setting.
* Tilting is another special case.
* Known examples include
  - trees of rational curves with nodal singularities
  - toric surfaces S with $Br(S)=0$
  - certain non-factorial nodal Fano 3-folds (see also Evgeny's talk later in this workshop)
  - Can produce further examples from these by using blow-ups

* If X has KSOD, then $K_{-1}(X) = 0$ equiv. $D_{sg}(X)$ is idempotent complete.
This obstruction can be used to explain observations by Kawamata and rule out KSODs in further examples

* If $\dim(X)$ is odd & X has KSOD then we can use $D_{sg}(X)$ and cluster theory to restrict the analytic types of singularities of X to type A_1-hypersurfaces (= nodal singularities) in many cases.