Spherical subcategories and new invariants for triangulated categories

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CRM
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This is joint work with A. Hochenegger & D. Ploog:

Spherical subcategories in algebraic geometry, arXiv:1208.4046.
Spherical subcategories in representation theory, arXiv:1502.06838.

Setup

$\mathcal{T}$ 'nice' triangulated category, for example

- $\mathcal{D}^b(A) := \mathcal{D}^b(A - \mathrm{mod})$, A fin. dim. k-algebra with gl.dim$A < \infty$ or
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More precisely, today we assume that $\mathcal{T}$ is

- k-linear (where k is an algebraically closed field);
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  $$\mathrm{Hom}_{\mathcal{T}}(A, B) \cong \mathrm{Hom}_{\mathcal{T}}(B, \mathbb{S}A)^*$$

  are binatural isomorphisms). For example
  - If $\mathcal{T} = \mathcal{D}^b(A)$ (gl.dim$A < \infty$), then $\mathbb{S} \cong A^* \otimes^{\mathbb{L}}_A -$ (Nakayama functor)
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