Relative Singularity Categories

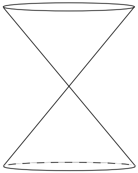
Martin Kalck

Bielefeld University, Germany

Schwerpunkttagung SPP 1388
Bad Boll
March 28, 2013

Motivation and Overview

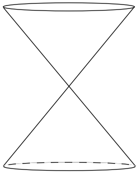
Singularity



$X = \mathbb{C}^2/\mathbb{Z}_2$

Motivation and Overview

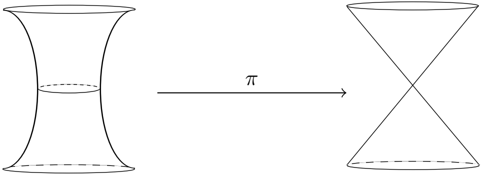
Singularity



$$X = \mathbb{C}^2/\mathbb{Z}_2$$
$$= V(X^2 + Y^2 + Z^2) \subseteq \mathbb{A}^3$$

Motivation and Overview

Resolution | Singularity



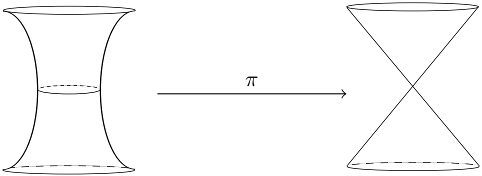
Y

$$X = \mathbb{C}^2/\mathbb{Z}_2$$
$$= V(X^2 + Y^2 + Z^2) \subseteq \mathbb{A}^3$$

Motivation and Overview

Resolution

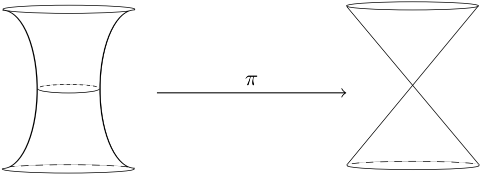
Singularity



$\mathsf{Perf}(X)$

Motivation and Overview

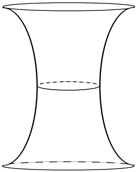
Resolution　　　　　　　　　　Singularity



$$\mathsf{Perf}(X) \underset{\text{'smooth'}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\, X)$$

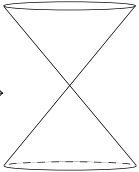
Motivation and Overview

Resolution



$\xrightarrow{\pi}$

Singularity

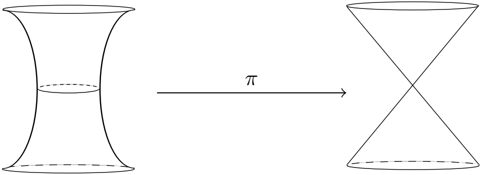


$\mathcal{D}^b(\mathsf{Coh}\, Y)$

$\mathsf{Perf}(X) \underset{\text{'smooth'}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\, X)$

Motivation and Overview

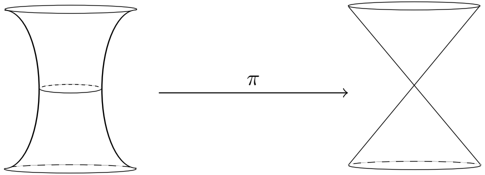
Resolution Singularity



$$\mathcal{D}^b(\mathsf{Coh}\, Y) \underset{\text{‘categorical resolution’}}{\overset{\mathbb{L}\pi^*}{\longhookleftarrow}} \mathsf{Perf}(X) \underset{\text{‘smooth’}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\, X)$$

Motivation and Overview

Resolution Singularity



$$\mathcal{D}^b(\mathrm{Coh}\,Y) \underset{\text{‘categorical resolution’}}{\xleftarrow{\ \mathbb{L}\pi^*\ }} \mathrm{Perf}(X) \underset{\text{‘smooth’}}{\subseteq} \mathcal{D}^b(\mathrm{Coh}\,X)$$

Aim

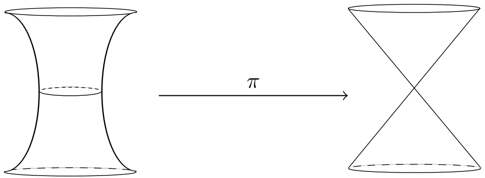
Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathrm{Coh}\,Y)}{\mathrm{Perf}(X)}$$

relative singularity category

Motivation and Overview

Resolution | Singularity



$$\mathcal{D}^b(\mathrm{Coh}\,Y) \underset{\text{‘categorical resolution’}}{\overset{\mathbb{L}\pi^*}{\hookleftarrow}} \mathrm{Perf}(X) \underset{\text{‘smooth’}}{\subseteq} \mathcal{D}^b(\mathrm{Coh}\,X)$$

Aim

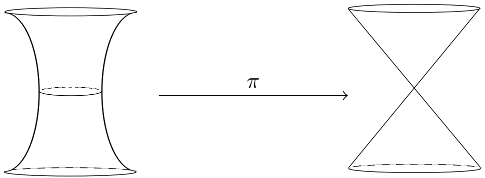
Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathrm{Coh}\,Y)}{\mathrm{Perf}(X)} \qquad \textit{and} \qquad \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathrm{Coh}\,X)}{\mathrm{Perf}(X)}$$

relative singularity category | *singularity category*

Motivation and Overview

Resolution ⟶ 1 Singularities



$$\mathcal{D}^b(\mathsf{Coh}\, Y) \xleftarrow[\text{‘categorical resolution’}]{\mathbb{L}\pi^*} \mathsf{Perf}(X) \underset{\text{‘smooth’}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\, X)$$

Aim

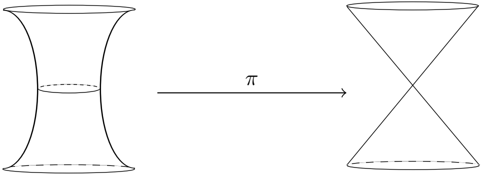
Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathsf{Coh}\, Y)}{\mathsf{Perf}(X)} \qquad \textit{and} \qquad \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathsf{Coh}\, X)}{\mathsf{Perf}(X)}$$

relative singularity category — *singularity category*

Motivation and Overview

Resolution   1 Singularities



$$\mathcal{D}^b(\mathrm{Coh}\,Y) \underset{\text{‘categorical resolution’}}{\xhookleftarrow{\mathbb{L}\pi^*}} \mathrm{Perf}(X) \underset{\text{‘smooth’}}{\subseteq} \mathcal{D}^b(\mathrm{Coh}\,X)$$

Aim

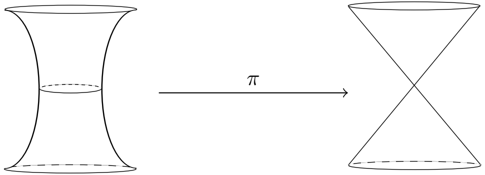
Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathrm{Coh}\,Y)}{\mathrm{Perf}(X)} \qquad \textit{and} \qquad \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathrm{Coh}\,X)}{\mathrm{Perf}(X)}$$

relative singularity category   2 *Singularity categories*

Motivation and Overview

3 Resolutions   1 Singularities



$$\mathcal{D}^b(\mathsf{Coh}\,Y) \xleftarrow[\text{'categorical resolution'}]{\mathbb{L}\pi^*} \mathsf{Perf}(X) \underset{\text{'smooth'}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\,X)$$

Aim

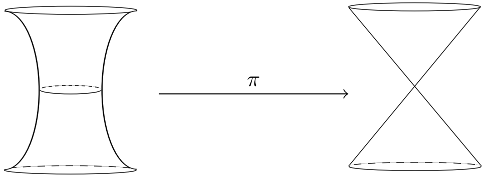
Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathsf{Coh}\,Y)}{\mathsf{Perf}(X)} \qquad \textit{and} \qquad \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathsf{Coh}\,X)}{\mathsf{Perf}(X)}$$

relative singularity category   2 *Singularity categories*

Motivation and Overview

3 Resolutions 1 Singularities



$$\mathcal{D}^b(\mathrm{Coh}\, Y) \underset{\text{‘categorical resolution’}}{\xleftarrow{\ \mathbb{L}\pi^*\ }} \mathrm{Perf}(X) \underset{\text{‘smooth’}}{\subseteq} \mathcal{D}^b(\mathrm{Coh}\, X)$$

Aim

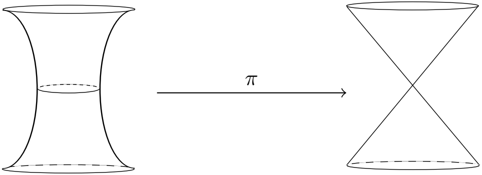
Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathrm{Coh}\, Y)}{\mathrm{Perf}(X)} \qquad \textit{and} \qquad \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathrm{Coh}\, X)}{\mathrm{Perf}(X)}$$

4 *Relative singularity categories* 2 *Singularity categories*

Motivation and Overview

3 Resolutions 1 Singularities



$$\mathcal{D}^b(\mathsf{Coh}\,Y) \xleftarrow[\text{`categorical resolution'}]{\mathbb{L}\pi^*} \mathsf{Perf}(X) \underset{\text{`smooth'}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\,X)$$

Aim

Use representation theory to describe the triangulated quotients

5 *Relations*

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathsf{Coh}\,Y)}{\mathsf{Perf}(X)} \longleftrightarrow \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathsf{Coh}\,X)}{\mathsf{Perf}(X)}$$

4 *Relative singularity categories* 2 *Singularity categories*

1. Singularities

Definition/Theorem (Auslander, Buchsbaum & Serre)

Let R be a commutative Noetherian ring of finite Krull dimension.

Definition/Theorem (Auslander, Buchsbaum & Serre)

Let R be a commutative Noetherian ring of finite Krull dimension. Then

$\text{Spec}(R)$ *is* **regular**

Definition/Theorem (Auslander, Buchsbaum & Serre)

Let R be a commutative Noetherian ring of finite Krull dimension. Then

$$\operatorname{Spec}(R) \text{ is } \textbf{regular} \;\Leftrightarrow\; \operatorname{gl.dim}(R) < \infty$$

Definition/Theorem (Auslander, Buchsbaum & Serre)

Let R be a commutative Noetherian ring of finite Krull dimension. Then

$$\mathsf{Spec}(R) \textit{ is } \textbf{regular} \;\Leftrightarrow\; \mathsf{kr.dim}(R) = \mathsf{gl.dim}(R) < \infty$$

Definition/Theorem (Auslander, Buchsbaum & Serre)

Let R be a commutative Noetherian ring of finite Krull dimension. Then

$$\operatorname{Spec}(R) \textit{ is } \textbf{regular} \Leftrightarrow \operatorname{kr.dim}(R) = \operatorname{gl.dim}(R) < \infty$$

This suggests the following definition for (non-commutative) rings R:

1. Singularities

Definition/Theorem (Auslander, Buchsbaum & Serre)

Let R be a commutative Noetherian ring of finite Krull dimension. Then

$$\mathsf{Spec}(R) \textit{ is } \textbf{regular} \;\Leftrightarrow\; \operatorname{kr.dim}(R) = \operatorname{gl.dim}(R) < \infty$$

This suggests the following definition for (non-commutative) rings R:

Definition

- R is **regular/smooth** $:\Leftrightarrow \operatorname{gl.dim}(R) < \infty$;

Definition/Theorem (Auslander, Buchsbaum & Serre)

Let R be a commutative Noetherian ring of finite Krull dimension. Then

$$\mathsf{Spec}(R) \textit{ is } \textbf{regular} \;\Leftrightarrow\; \mathrm{kr.\,dim}(R) = \mathrm{gl.\,dim}(R) < \infty$$

This suggests the following definition for (non-commutative) rings R:

Definition

- R is **regular/smooth** $:\Leftrightarrow \mathrm{gl.\,dim}(R) < \infty$;
- R is **singular** $:\Leftrightarrow \mathrm{gl.\,dim}(R) = \infty$.

1. Singularities (Examples)

Example

Let k be a field.

(i) For $R = k[x]/(x^2)$ the simple module $k = R/(x)$ has infinite projective dimension

1. Singularities (Examples)

Example

Let k be a field.

(i) For $R = k[x]/(x^2)$ the simple module $k = R/(x)$ has infinite projective dimension ($\cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \to R/(x) \to 0$)

1. Singularities (Examples)

Example

Let k be a field.

(i) For $R = k[x]/(x^2)$ the simple module $k = R/(x)$ has infinite projective dimension ($\cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \to R/(x) \to 0$)
$\Rightarrow$ R is **singular**.

(ii) More generally, let A be a finite-dimensional **selfinjective** k-algebra

1. Singularities (Examples)

Example

Let k be a field.

(i) For $R = k[x]/(x^2)$ the simple module $k = R/(x)$ has infinite projective dimension ($\cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \to R/(x) \to 0$)
$\Rightarrow R$ is **singular**.

(ii) More generally, let A be a finite-dimensional **selfinjective** k-algebra (i.e. A is an injective A-module).

1. Singularities (Examples)

Example

Let k be a field.

(i) For $R = k[x]/(x^2)$ the simple module $k = R/(x)$ has infinite projective dimension ($\cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \to R/(x) \to 0$)
$\Rightarrow R$ is **singular**.

(ii) More generally, let A be a finite-dimensional **selfinjective** k-algebra (i.e. A is an injective A-module).For example, Frobenius algebras, symmetric algebras, Hopf algebras and group algebras are selfinjective.

1. Singularities (Examples)

Example

Let k be a field.

(i) For $R = k[x]/(x^2)$ the simple module $k = R/(x)$ has infinite projective dimension ($\cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \to R/(x) \to 0$) $\Rightarrow R$ is **singular**.

(ii) More generally, let A be a finite-dimensional **selfinjective** k-algebra (i.e. A is an injective A-module).For example, Frobenius algebras, symmetric algebras, Hopf algebras and group algebras are selfinjective.

Lemma: $\operatorname{gl.dim}(A) < \infty \Leftrightarrow A$ is semi-simple.

1. Singularities (Examples)

(iii) Let $S = \mathbb{C}[\![x_0, \ldots, x_d]\!]$, $\mathfrak{m} = (x_0, \ldots, x_d) \subseteq S$ and $f \in \mathfrak{m}$.

1. Singularities (Examples)

(iii) Let $S = \mathbb{C}[\![x_0, \ldots, x_d]\!]$, $\mathfrak{m} = (x_0, \ldots, x_d) \subseteq S$ and $f \in \mathfrak{m}$. Then

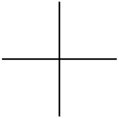
the **hypersurface ring** $S/(f)$ is singular $\Leftrightarrow 0 \neq f \in \mathfrak{m}^2$

1. Singularities (Examples)

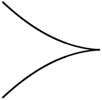
(iii) Let $S = \mathbb{C}[\![x_0, \dots, x_d]\!]$, $\mathfrak{m} = (x_0, \dots, x_d) \subseteq S$ and $f \in \mathfrak{m}$. Then

the **hypersurface ring** $S/(f)$ is singular $\Leftrightarrow 0 \neq f \in \mathfrak{m}^2$

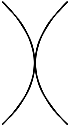
Some curve singularities



$f = xy$



$f = x^3 - y^2$



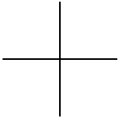
$f = x^4 - y^2$

1. Singularities (Examples)

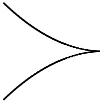
(iii) Let $S = \mathbb{C}[\![x_0, \ldots, x_d]\!]$, $\mathfrak{m} = (x_0, \ldots, x_d) \subseteq S$ and $f \in \mathfrak{m}$. Then

the **hypersurface ring** $S/(f)$ is singular $\Leftrightarrow 0 \neq f \in \mathfrak{m}^2$

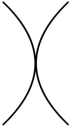
Some curve singularities



$f = xy$

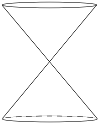


$f = x^3 - y^2$

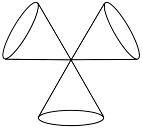


$f = x^4 - y^2$

Some surface singularities



$f = x^2 + y^2 + z^2$

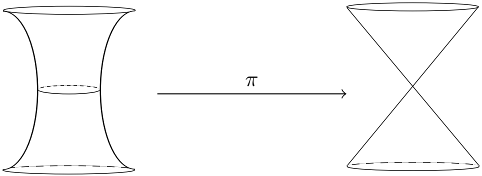


$f = x^2y + y^3 + z^2$

Overview

3 Resolutions

1 Singularities



$$\mathcal{D}^b(\mathsf{Coh}\, Y) \xleftarrow[\text{‘categorical resolution’}]{\mathbb{L}\pi^*} \mathsf{Perf}(X) \underset{\text{‘smooth’}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\, X)$$

Aim

Use representation theory to describe the triangulated quotients

5 *Relations*

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathsf{Coh}\, Y)}{\mathsf{Perf}(X)} \longleftrightarrow \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathsf{Coh}\, X)}{\mathsf{Perf}(X)}$$

4 *Relative singularity categories*

2 *Singularity categories*

2. Singularity categories

Definition/Theorem

Let R be a Noetherian ring. Then the **bounded derived category** $\mathcal{D}^b(R)$ *of* R

2. Singularity categories

Definition/Theorem

Let R be a Noetherian ring. Then the **bounded derived category** $\mathcal{D}^b(R)$ *of R is equivalent to the homotopy category $K^{-,b}(\operatorname{proj} -R)$.*

Definition *of $K^{-,b}(\operatorname{proj} -R)$:*

2. Singularity categories

Definition/Theorem

Let R be a Noetherian ring. Then the **bounded derived category** $\mathcal{D}^b(R)$ *of R is equivalent to the homotopy category $K^{-,b}(\operatorname{proj} -R)$.*

Definition *of $K^{-,b}(\operatorname{proj} -R)$:*
Objects: *Right bounded complexes P^* of projective R-modules,*

2. Singularity categories

Definition/Theorem

Let R be a Noetherian ring. Then the **bounded derived category** $\mathcal{D}^b(R)$ *of R is equivalent to the homotopy category $K^{-,b}(\operatorname{proj} - R)$.*

Definition *of $K^{-,b}(\operatorname{proj} - R)$:*
Objects: *Right bounded complexes P^* of projective R-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots$$

2. Singularity categories

Definition/Theorem

Let R be a Noetherian ring. Then the **bounded derived category** $\mathcal{D}^b(R)$ *of R is equivalent to the homotopy category $K^{-,b}(\mathrm{proj} - R)$.*

Definition *of $K^{-,b}(\mathrm{proj} - R)$:*
Objects: *Right bounded complexes P^* of projective R-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots \quad d_i d_{i+1} = 0$$

2. Singularity categories

Definition/Theorem

Let R be a Noetherian ring. Then the **bounded derived category** $\mathcal{D}^b(R)$ *of R is equivalent to the homotopy category $K^{-,b}(\operatorname{proj} - R)$.*

Definition *of $K^{-,b}(\operatorname{proj} - R)$:*
Objects: *Right bounded complexes P^* of projective R-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots \quad d_i d_{i+1} = 0$$

with **bounded cohomology**,

2. Singularity categories

Definition/Theorem

Let R be a Noetherian ring. Then the **bounded derived category** $\mathcal{D}^b(R)$ *of R is equivalent to the homotopy category $K^{-,b}(\operatorname{proj} - R)$.*

Definition *of $K^{-,b}(\operatorname{proj} - R)$:*
Objects: *Right bounded complexes P^* of projective R-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots \quad d_i d_{i+1} = 0$$

with **bounded cohomology**, *i.e.* $H^i(P^*) := \ker d_i / \operatorname{im} d_{i+1} = 0$ *for* $i \gg 0$.

2. Singularity categories

Definition/Theorem

Let R be a Noetherian ring. Then the **bounded derived category** $\mathcal{D}^b(R)$ *of R is equivalent to the homotopy category $K^{-,b}(\operatorname{proj} -R)$.*

Definition *of $K^{-,b}(\operatorname{proj} -R)$:*
Objects: *Right bounded complexes P^* of projective R-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots \quad d_i d_{i+1} = 0$$

with **bounded cohomology**, *i.e.* $H^i(P^*) := \ker d_i / \operatorname{im} d_{i+1} = 0$ *for* $i \gg 0$.
In other words, these are **projective resolutions** *of* **bounded complexes**.

Definition/Theorem

Definition *of* $K^{-,b}(\operatorname{proj} -R)$*:*
Objects: *Right bounded complexes* P^* *of projective* R*-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots \quad d_i d_{i+1} = 0$$

with **bounded cohomology**, *i.e.* $H^i(P^*) := \ker d_i / \operatorname{im} d_{i+1} = 0$ *for* $i \gg 0$.
In other words, these are projective resolutions of bounded complexes.

Morphisms:

2. Singularity categories

Definition/Theorem

Definition *of* $K^{-,b}(\operatorname{proj} -R)$*:*

Objects: *Right bounded complexes* P^* *of projective* R*-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots \quad d_i d_{i+1} = 0$$

with **bounded cohomology**, *i.e.* $H^i(P^*) := \ker d_i / \operatorname{im} d_{i+1} = 0$ *for* $i \gg 0$.

In other words, these are projective resolutions of bounded complexes.

Morphisms: *morphisms of complexes* f^*,

2. Singularity categories

Definition/Theorem

Definition *of* $K^{-,b}(\mathrm{proj}-R)$*:*

Objects: *Right bounded complexes* P^* *of projective* R*-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots \quad d_i d_{i+1} = 0$$

with **bounded cohomology***, i.e.* $H^i(P^*) := \ker d_i / \operatorname{im} d_{i+1} = 0$ *for* $i \gg 0$.

In other words, these are projective resolutions of bounded complexes.

Morphisms: *morphisms of complexes* f^**, i.e.*

$$\begin{tikzcd}
\cdots \arrow[r, "d_{t+1}"] \arrow[d, "f_{t+1}"'] \arrow[dr, phantom, "\circlearrowright"] & P_t \arrow[r, "d_t"] \arrow[d, "f_t"] \arrow[dr, phantom, "\circlearrowright"] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "f_{t-1}"] & \cdots \arrow[r, "d_{j+1}"] \arrow[d, phantom, "\cdots"] & P_j \arrow[r, "d_j"] \arrow[d, "f_j"] & \cdots \\
\cdots \arrow[r, "\partial_{t+1}"] & Q_t \arrow[r, "\partial_t"] & Q_{t-1} \arrow[r, "\partial_{t-1}"] & \cdots \arrow[r, "\partial_{j+1}"] & Q_j \arrow[r, "\partial_j"] & \cdots
\end{tikzcd}$$

2. Singularity categories

Definition/Theorem

Definition *of* $K^{-,b}(\operatorname{proj} - R)$*:*

Objects: *Right bounded complexes* P^* *of projective* R*-modules, i.e.*

$$P^* = \cdots \xrightarrow{d_{t+1}} P_t \xrightarrow{d_t} P_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_{j+1}} P_j \to 0 \to \cdots \quad d_i d_{i+1} = 0$$

with **bounded cohomology**, *i.e.* $H^i(P^*) := \ker d_i / \operatorname{im} d_{i+1} = 0$ *for* $i \gg 0$.

In other words, these are projective resolutions of bounded complexes.

Morphisms: *(equivalence classes of) morphisms of complexes* f^*, *i.e.*

$$\begin{tikzcd}
\cdots \arrow[r, "d_{t+1}"] \arrow[d, "f_{t+1}"'] \arrow[dr, phantom, "\circlearrowright"] & P_t \arrow[r, "d_t"] \arrow[d, "f_t"] \arrow[dr, phantom, "\circlearrowright"] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "f_{t-1}"] & \cdots \arrow[r, "d_{j+1}"] \arrow[d, phantom, "\cdots"] & P_j \arrow[r, "d_j"] \arrow[d, "f_j"] & \cdots \\
\cdots \arrow[r, "\partial_{t+1}"] & Q_t \arrow[r, "\partial_t"] & Q_{t-1} \arrow[r, "\partial_{t-1}"] & \cdots \arrow[r, "\partial_{j+1}"] & Q_j \arrow[r, "\partial_j"] & \cdots
\end{tikzcd}$$

2. Singularity categories

Example

Let $R = k[x]/(x^2)$. Then the indecomposable objects in $\mathcal{D}^b(R)$ are completely classified. Up to shift of complexes, they are given by:

2. Singularity categories

Example

Let $R = k[x]/(x^2)$. Then the indecomposable objects in $\mathcal{D}^b(R)$ are completely classified. Up to shift of complexes, they are given by:

$$\cdots \longrightarrow 0 \longrightarrow \overbrace{R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R}^{\mathbb{N} \ni l \text{ terms}} \longrightarrow 0 \longrightarrow \cdots \in K^b(\mathrm{proj} - R)$$

Example

Let $R = k[x]/(x^2)$. Then the indecomposable objects in $\mathcal{D}^b(R)$ are completely classified. Up to shift of complexes, they are given by:

$$\cdots \longrightarrow 0 \longrightarrow \overbrace{R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R}^{\mathbb{N} \ni l \text{ terms}} \longrightarrow 0 \longrightarrow \cdots \in K^b(\mathrm{proj} - R)$$

$$\cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \longrightarrow 0 \longrightarrow \cdots \in K^{-,b}(\mathrm{proj} - R)$$

2. Singularity categories

Example

Let $R = k[x]/(x^2)$. Then the indecomposable objects in $\mathcal{D}^b(R)$ are completely classified. Up to shift of complexes, they are given by:

$$
\begin{array}{c}
\cdots \longrightarrow 0 \longrightarrow \overbrace{R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R}^{\mathbb{N} \ni l \text{ terms}} \longrightarrow 0 \longrightarrow \cdots \in K^b(\mathrm{proj} - R) \\
\cap \\
\cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \longrightarrow 0 \longrightarrow \cdots \in K^{-,b}(\mathrm{proj} - R)
\end{array}
$$

2. Singularity categories

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**,

2. Singularity categories

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**, it consists of

- an equivalence $\Sigma\colon \mathcal{D}^b(R) \to \mathcal{D}^b(R)$

2. Singularity categories

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**, it consists of

- an equivalence $\Sigma\colon \mathcal{D}^b(R) \to \mathcal{D}^b(R)$ (the **shift functor**, given by shifting complexes)

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**, it consists of

- an equivalence $\Sigma\colon \mathcal{D}^b(R) \to \mathcal{D}^b(R)$ (the **shift functor**, given by shifting complexes)
- and a class of **(distinguished) triangles**, i.e. diagrams

$$X \to Y \to Z \to \Sigma(X)$$

with $X, Y, Z \in \mathcal{D}^b(R)$, satisfying certain axioms.

2. Singularity categories

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**, it consists of

- an equivalence $\Sigma\colon \mathcal{D}^b(R) \to \mathcal{D}^b(R)$ (the **shift functor**, given by shifting complexes)
- and a class of **(distinguished) triangles**, i.e. diagrams

$$X \to Y \to Z \to \Sigma(X)$$

  with $X, Y, Z \in \mathcal{D}^b(R)$, satisfying certain axioms.

We will make use of the following properties:

(i) Every $f\colon X \to Y$ yields a triangle $X \xrightarrow{f} Y \to C(f) \to \Sigma(X)$.

2. Singularity categories

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**, it consists of

- an equivalence $\Sigma\colon \mathcal{D}^b(R) \to \mathcal{D}^b(R)$ (the **shift functor**, given by shifting complexes)
- and a class of **(distinguished) triangles**, i.e. diagrams

$$X \to Y \to Z \to \Sigma(X)$$

  with $X, Y, Z \in \mathcal{D}^b(R)$, satisfying certain axioms.

We will make use of the following properties:

(i) Every $f\colon X \to Y$ yields a triangle $X \xrightarrow{f} Y \to C(f) \to \Sigma(X)$.

(ii) $C(f) \cong 0 \;\Leftrightarrow\; f$ is an isomorphism.

2. Singularity categories

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**, it consists of

- an equivalence $\Sigma\colon \mathcal{D}^b(R) \to \mathcal{D}^b(R)$ (the **shift functor**, given by shifting complexes)
- and a class of **(distinguished) triangles**, i.e. diagrams

$$X \to Y \to Z \to \Sigma(X)$$

  with $X, Y, Z \in \mathcal{D}^b(R)$, satisfying certain axioms.

We will make use of the following properties:

(i) Every $f\colon X \to Y$ yields a triangle $X \xrightarrow{f} Y \to C(f) \to \Sigma(X)$.

(ii) $C(f) \cong 0 \;\Leftrightarrow\; f$ is an isomorphism.
In particular, $X \xrightarrow{\mathrm{id}} X \to 0 \to \Sigma(X)$ is a triangle.

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**, it consists of

- an equivalence $\Sigma\colon \mathcal{D}^b(R) \to \mathcal{D}^b(R)$ (the **shift functor**, given by shifting complexes)
- and a class of **(distinguished) triangles**, i.e. diagrams

$$X \to Y \to Z \to \Sigma(X)$$

  with $X, Y, Z \in \mathcal{D}^b(R)$, satisfying certain axioms.

We will make use of the following properties:

(i) Every $f\colon X \to Y$ yields a triangle $X \xrightarrow{f} Y \to C(f) \to \Sigma(X)$.

(ii) $C(f) \cong 0 \;\Leftrightarrow\; f$ is an isomorphism.
In particular, $X \xrightarrow{\mathrm{id}} X \to 0 \to \Sigma(X)$ is a triangle.

(iii) The shift $Y \to Z \to \Sigma(X) \to \Sigma(Y)$ of a triangle, is a triangle.

2. Singularity categories

Remark

The derived category $\mathcal{D}^b(R)$ has a **triangulated structure**, it consists of

- an equivalence $\Sigma\colon \mathcal{D}^b(R) \to \mathcal{D}^b(R)$ (the **shift functor**, given by shifting complexes)
- and a class of **(distinguished) triangles**, i.e. diagrams

$$X \to Y \to Z \to \Sigma(X)$$

  with $X, Y, Z \in \mathcal{D}^b(R)$, satisfying certain axioms.

We will make use of the following properties:

(i) Every $f\colon X \to Y$ yields a triangle $X \xrightarrow{f} Y \to C(f) \to \Sigma(X)$.

(ii) $C(f) \cong 0 \;\Leftrightarrow\; f$ is an isomorphism.
In particular, $X \xrightarrow{\mathrm{id}} X \to 0 \to \Sigma(X)$ is a triangle.

(iii) The shift $Y \to Z \to \Sigma(X) \to \Sigma(Y)$ of a triangle, is a triangle.
In particular, $0 \to X \xrightarrow{\mathrm{id}} X \to 0$ is a triangle.

Remark

(iv) The short exact sequence of complexes

Remark

(iv) The short exact sequence of complexes

$$\begin{tikzcd}
X = \cdots \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & P_t \arrow[r, "d_t"] \arrow[d, "\mathrm{id}"'] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "d_{s+1}"] & P_s \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Y = \cdots \arrow[r, "d_{t+3}"] & P_{t+2} \arrow[r, "d_{t+2}"] & P_{t+1} \arrow[r, "d_{t+1}"] & P_t \arrow[r, "d_t"] & P_{t-1} \arrow[r, "d_{t-1}"] & \cdots \arrow[r, "d_{s+1}"] & P_s \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Remark

(iv) The short exact sequence of complexes

$$\begin{tikzcd}
X = \cdots \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & P_t \arrow[r, "d_t"] \arrow[d, "\mathrm{id}"'] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "d_{s+1}"] & P_s \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Y = \cdots \arrow[r, "d_{t+3}"] \arrow[d, "p"'] & P_{t+2} \arrow[r, "d_{t+2}"] \arrow[d, "\mathrm{id}"'] & P_{t+1} \arrow[r, "d_{t+1}"] \arrow[d, "\mathrm{id}"'] & P_t \arrow[r, "d_t"] \arrow[d, "0"'] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "0"'] & \cdots \arrow[r, "d_{s+1}"] & P_s \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Z = \cdots \arrow[r, "d_{t+3}"] & P_{t+2} \arrow[r, "d_{t+2}"] & P_{t+1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Remark

(iv) The short exact sequence of complexes

$$\begin{tikzcd}
X = \cdots \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & P_t \arrow[r, "d_t"] \arrow[d, "\mathrm{id}"'] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "d_{s+1}"] & P_s \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Y = \cdots \arrow[r, "d_{t+3}"] \arrow[d, "p"'] & P_{t+2} \arrow[r, "d_{t+2}"] \arrow[d, "\mathrm{id}"'] & P_{t+1} \arrow[r, "d_{t+1}"] \arrow[d, "\mathrm{id}"'] & P_t \arrow[r, "d_t"] \arrow[d, "0"'] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "0"'] & \cdots \arrow[r, "d_{s+1}"] & P_s \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Z = \cdots \arrow[r, "d_{t+3}"] & P_{t+2} \arrow[r, "d_{t+2}"] & P_{t+1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

yields a triangle $X \xrightarrow{i} Y \xrightarrow{p} Z \to \Sigma(X)$ in $\mathcal{D}^b(R)$.

Remark

(iv) The short exact sequence of complexes

$$\begin{tikzcd}
X = \cdots \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & P_t \arrow[r, "d_t"] \arrow[d, "\mathrm{id}"'] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "d_{s+1}"] & P_s \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Y = \cdots \arrow[r, "d_{t+3}"] \arrow[d, "p"'] & P_{t+2} \arrow[r, "d_{t+2}"] \arrow[d, "\mathrm{id}"'] & P_{t+1} \arrow[r, "d_{t+1}"] \arrow[d, "\mathrm{id}"'] & P_t \arrow[r, "d_t"] \arrow[d, "0"'] & P_{t-1} \arrow[r, "d_{t-1}"] \arrow[d, "0"'] & \cdots \arrow[r, "d_{s+1}"] & P_s \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Z = \cdots \arrow[r, "d_{t+3}"] & P_{t+2} \arrow[r, "d_{t+2}"] & P_{t+1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

yields a triangle $X \xrightarrow{i} Y \xrightarrow{p} Z \to \Sigma(X)$ in $\mathcal{D}^b(R)$.

(v) A **triangulated subcategory** $\mathcal{U}$ of $\mathcal{D}^b(R)$ is a full subcategory, s.th. $\Sigma\colon \mathcal{U} \xrightarrow{\sim} \mathcal{U}$ and $C(f) \in \mathcal{U}$ for every morphism $f\colon X \to Y$ in $\mathcal{U}$.

2. Singularity categories

Definition

Let $\mathcal{U} \subseteq \mathcal{D}^b(R)$ be a triangulated subcategory.

2. Singularity categories

Definition

Let $\mathcal{U} \subseteq \mathcal{D}^b(R)$ be a triangulated subcategory. The **triangulated quotient category** $\mathcal{D}^b(R)/\mathcal{U}$ has the same **objects** as $\mathcal{D}^b(R)$.

2. Singularity categories

Definition

Let $\mathcal{U} \subseteq \mathcal{D}^b(R)$ be a triangulated subcategory. The **triangulated quotient category** $\mathcal{D}^b(R)/\mathcal{U}$ has the same **objects** as $\mathcal{D}^b(R)$.
We will need the following property of morphisms in $\mathcal{D}^b(R)/\mathcal{U}$:

2. Singularity categories

Definition

Let $\mathcal{U} \subseteq \mathcal{D}^b(R)$ be a triangulated subcategory. The **triangulated quotient category** $\mathcal{D}^b(R)/\mathcal{U}$ has the same **objects** as $\mathcal{D}^b(R)$.
We will need the following property of morphisms in $\mathcal{D}^b(R)/\mathcal{U}$:
If $X \xrightarrow{f} Y \to Z \to \Sigma(X)$ is a **triangle** in $\mathcal{D}^b(R)$ such that $Z \in \mathcal{U}$, then f is an **isomorphism** in $\mathcal{D}^b(R)/\mathcal{U}$.

2. Singularity categories

Definition

Let $\mathcal{U} \subseteq \mathcal{D}^b(R)$ be a triangulated subcategory. The **triangulated quotient category** $\mathcal{D}^b(R)/\mathcal{U}$ has the same **objects** as $\mathcal{D}^b(R)$.
We will need the following property of morphisms in $\mathcal{D}^b(R)/\mathcal{U}$:
If $X \xrightarrow{f} Y \to Z \to \Sigma(X)$ is a **triangle** in $\mathcal{D}^b(R)$ such that $Z \in \mathcal{U}$, then f is an **isomorphism** in $\mathcal{D}^b(R)/\mathcal{U}$.

Definition

Let R be a Noetherian ring. The **singularity category** of R is the quotient category

$$\mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(R)}{K^b(\mathrm{proj}-R)}$$

2. Singularity categories

Definition

Let R be a Noetherian ring. The **singularity category** of R is the quotient category

$$\mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(R)}{K^b(\operatorname{proj} - R)}$$

2. Singularity categories

Definition

Let R be a Noetherian ring. The **singularity category** of R is the quotient category

$$\mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(R)}{K^b(\mathrm{proj} - R)}$$

Remark

- R is regular (i.e. $\mathrm{gl.dim}(R) < \infty$) $\Leftrightarrow K^b(\mathrm{proj} - R) \cong \mathcal{D}^b(R)$.

2. Singularity categories

Definition

Let R be a Noetherian ring. The **singularity category** of R is the quotient category

$$\mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(R)}{K^b(\mathrm{proj}-R)}$$

Remark

- R is regular (i.e. $\mathrm{gl.\,dim}(R) < \infty$) $\Leftrightarrow K^b(\mathrm{proj}-R) \cong \mathcal{D}^b(R)$.
- In particular, $\mathcal{D}_{sg}(R) = 0$ in this case.

2. Singularity categories

Definition

Let R be a Noetherian ring. The **singularity category** of R is the quotient category

$$\mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(R)}{K^b(\mathrm{proj} - R)}$$

Remark

- R is regular (i.e. $\mathrm{gl.dim}(R) < \infty$) $\Leftrightarrow K^b(\mathrm{proj} - R) \cong \mathcal{D}^b(R)$.
- In particular, $\mathcal{D}_{sg}(R) = 0$ in this case.
- Moreover, this suggests to view $K^b(\mathrm{proj} - R) \subseteq \mathcal{D}^b(R)$ as the **'smooth part'**

2. Singularity categories

Definition

Let R be a Noetherian ring. The **singularity category** of R is the quotient category

$$\mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(R)}{K^b(\mathrm{proj}-R)}$$

Remark

- R is regular (i.e. gl. dim$(R) < \infty$) $\Leftrightarrow K^b(\mathrm{proj}-R) \cong \mathcal{D}^b(R)$.
- In particular, $\mathcal{D}_{sg}(R) = 0$ in this case.
- Moreover, this suggests to view $K^b(\mathrm{proj}-R) \subseteq \mathcal{D}^b(R)$ as the **'smooth part'** and $\mathcal{D}_{sg}(R)$ as a **measure for the complexity of the singularities** of $\mathrm{Spec}(R)$.

2. Singularity categories

Example

Let $R = k[x]/(x^2)$. We determine the indecomposables in $\mathcal{D}_{sg}(R)$.

2. Singularity categories

Example

Let $R = k[x]/(x^2)$. We determine the indecomposables in $\mathcal{D}_{sg}(R)$. Recall,

$$\begin{array}{c} \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \longrightarrow 0 \longrightarrow \cdots \in K^b(\operatorname{proj} - R) \\ \cap \\ \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \longrightarrow 0 \longrightarrow \cdots \in K^{-,b}(\operatorname{proj} - R) \end{array}$$

are the indecomposables in $\mathcal{D}^b(R)$ (up to shift).

2. Singularity categories

Example

Let $R = k[x]/(x^2)$. We determine the indecomposables in $\mathcal{D}_{sg}(R)$. Recall,

$$\begin{array}{c} \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \longrightarrow 0 \longrightarrow \cdots \in K^b(\mathrm{proj}-R) \\ \bigcap \\ \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} \cdots \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \longrightarrow 0 \longrightarrow \cdots \in K^{-,b}(\mathrm{proj}-R) \end{array}$$

are the indecomposables in $\mathcal{D}^b(R)$ (up to shift). The bounded complexes vanish in the singularity category.

Example

Consider the following exact sequence of complexes

2. Singularity categories

Example

Consider the following exact sequence of complexes

$$
\begin{tikzcd}
X = \cdots \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] \arrow[rrr, phantom, "\overbrace{\hspace{5cm}}^{n \text{ terms}}", shift left=4ex] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "x\cdot"] & R \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Y = \cdots \arrow[r, "x\cdot"] \arrow[d, "p"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "0"'] & R \arrow[r, "x\cdot"] \arrow[d, "0"'] & \cdots \arrow[r, "x\cdot"] & R \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\Sigma^n(Y) = \cdots \arrow[r, "x\cdot"] & R \arrow[r, "x\cdot"] & R \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

2. Singularity categories

Example

Consider the following exact sequence of complexes

n terms

$$
\begin{tikzcd}
X = \cdots \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "x\cdot"] & R \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Y = \cdots \arrow[r, "x\cdot"] \arrow[d, "p"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "0"'] & R \arrow[r, "x\cdot"] \arrow[d, "0"'] & \cdots \arrow[r, "x\cdot"] & R \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\Sigma^n(Y) = \cdots \arrow[r, "x\cdot"] & R \arrow[r, "x\cdot"] & R \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$,

Example

Consider the following exact sequence of complexes

$$
\begin{tikzcd}
 & & & & & \overbrace{\hspace{6cm}}^{n \text{ terms}} & & & & \\
X = \cdots \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "x\cdot"] & R \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Y = \cdots \arrow[r, "x\cdot"] \arrow[d, "p"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "0"'] & R \arrow[r, "x\cdot"] \arrow[d, "0"'] & \cdots \arrow[r, "x\cdot"] & R \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\Sigma^n(Y) = \cdots \arrow[r, "x\cdot"] & R \arrow[r, "x\cdot"] & R \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$, which we may shift to obtain

$$Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X) \xrightarrow{\Sigma(i)} \Sigma(Y)$$

2. Singularity categories

Example

This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$, which we may shift to obtain a triangle

$$Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X) \xrightarrow{\Sigma(i)} \Sigma(Y)$$

Since $\Sigma(X) \in K^b(\mathrm{proj} - R)$, we see that $p\colon Y \to \Sigma^n(Y)$ is an isomorphism in the singularity category.

2. Singularity categories

Example

This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$, which we may shift to obtain a triangle

$$Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X) \xrightarrow{\Sigma(i)} \Sigma(Y)$$

Since $\Sigma(X) \in K^b(\mathrm{proj} - R)$, we see that $p\colon Y \to \Sigma^n(Y)$ is an isomorphism in the singularity category. This shows:

- all shifts of Y are isomorphic in $\mathcal{D}_{sg}(R)$.

2. Singularity categories

Example

This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$, which we may shift to obtain a triangle

$$Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X) \xrightarrow{\Sigma(i)} \Sigma(Y)$$

Since $\Sigma(X) \in K^b(\mathrm{proj} - R)$, we see that $p\colon Y \to \Sigma^n(Y)$ is an isomorphism in the singularity category. This shows:

- all shifts of Y are isomorphic in $\mathcal{D}_{sg}(R)$. In other words, there is precisely one indecomposable object in $\mathcal{D}_{sg}(R)$.

2. Singularity categories

Example

This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$, which we may shift to obtain a triangle

$$Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X) \xrightarrow{\Sigma(i)} \Sigma(Y)$$

Since $\Sigma(X) \in K^b(\mathrm{proj}-R)$, we see that $p\colon Y \to \Sigma^n(Y)$ is an isomorphism in the singularity category. This shows:

- all shifts of Y are isomorphic in $\mathcal{D}_{sg}(R)$. In other words, there is precisely one indecomposable object in $\mathcal{D}_{sg}(R)$.
- Moreover, the endomorphism algebra of Y in $\mathcal{D}_{sg}(R)$ is isomorphic to k.

Example

This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$, which we may shift to obtain a triangle

$$Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X) \xrightarrow{\Sigma(i)} \Sigma(Y)$$

Since $\Sigma(X) \in K^b(\operatorname{proj} -R)$, we see that $p\colon Y \to \Sigma^n(Y)$ is an isomorphism in the singularity category. This shows:

- all shifts of Y are isomorphic in $\mathcal{D}_{sg}(R)$. In other words, there is precisely one indecomposable object in $\mathcal{D}_{sg}(R)$.
- Moreover, the endomorphism algebra of Y in $\mathcal{D}_{sg}(R)$ is isomorphic to k. This gives an equivalence of additive categories

$$\mathcal{D}_{sg}(R) \cong \operatorname{mod} -k$$

2. Singularity categories

Remark

We can also consider the **stable category**

$$\underline{\mathrm{mod}} - R := \frac{\mathrm{mod}\, -R}{\mathrm{proj}\, -R}$$

2. Singularity categories

Remark

We can also consider the **stable category**

$$\underline{\mathrm{mod}} - R := \frac{\mathrm{mod} - R}{\mathrm{proj} - R}$$

- It is **triangulated** since R is self-injective, by work of Happel.

Remark

We can also consider the **stable category**

$$\underline{\mathrm{mod}} - R := \frac{\mathrm{mod} - R}{\mathrm{proj} - R}$$

- It is **triangulated** since R is self-injective, by work of Happel.
- Moreover, as an additive category

$$\underline{\mathrm{mod}} - R \cong \mathrm{mod} - k \cong \mathcal{D}_{sg}(R)$$

2. Singularity categories

Definition

Let R be two-sided Noetherian. R is called **(Iwanaga–)Gorenstein** ring if

$$\operatorname{inj.dim}_R R < \infty \text{ and } \operatorname{inj.dim} R_R < \infty.$$

2. Singularity categories

Definition

Let R be two-sided Noetherian. R is called **(Iwanaga–)Gorenstein** ring if

$$\text{inj.}\dim_R R < \infty \text{ and } \text{inj.}\dim R_R < \infty.$$

The category of **maximal Cohen–Macaulay** modules (MCM) is defined as

$$\mathrm{MCM}(R) = \{M \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(M, R) = 0 \text{ for all } i > 0\} \subseteq \mathrm{mod}-R.$$

2. Singularity categories

Definition

Let R be two-sided Noetherian. R is called **(Iwanaga–)Gorenstein** ring if

$$\text{inj.}\dim_R R < \infty \text{ and } \text{inj.}\dim R_R < \infty.$$

The category of **maximal Cohen–Macaulay** modules (MCM) is defined as

$$\mathrm{MCM}(R) = \{M \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(M, R) = 0 \text{ for all } i > 0\} \subseteq \mathrm{mod}-R.$$

Example

(i) **Regular** rings R are Gorenstein and $\mathrm{MCM}(R) = \mathrm{proj}-R$.

2. Singularity categories

Definition

Let R be two-sided Noetherian. R is called **(Iwanaga–)Gorenstein** ring if

$$\text{inj.}\dim_R R < \infty \text{ and } \text{inj.}\dim R_R < \infty.$$

The category of **maximal Cohen–Macaulay** modules (MCM) is defined as

$$\mathrm{MCM}(R) = \{M \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(M, R) = 0 \text{ for all } i > 0\} \subseteq \mathrm{mod}-R.$$

Example

(i) **Regular** rings R are Gorenstein and $\mathrm{MCM}(R) = \mathrm{proj}-R$.

(ii) **Self-injective** algebras are Gorenstein and $\mathrm{MCM}(R) = \mathrm{mod}-R$.

2. Singularity categories

Definition

Let R be two-sided Noetherian. R is called **(Iwanaga–)Gorenstein** ring if

$$\text{inj.}\dim_R R < \infty \text{ and } \text{inj.}\dim R_R < \infty.$$

The category of **maximal Cohen–Macaulay** modules (MCM) is defined as

$$\mathrm{MCM}(R) = \{M \in \mathrm{mod}-R \mid \mathrm{Ext}^i_R(M, R) = 0 \text{ for all } i > 0\} \subseteq \mathrm{mod}-R.$$

Example

(i) **Regular** rings R are Gorenstein and $\mathrm{MCM}(R) = \mathrm{proj}-R$.

(ii) **Self-injective** algebras are Gorenstein and $\mathrm{MCM}(R) = \mathrm{mod}-R$.

(iii) For **local commutative** Gorenstein rings R,

$$\mathrm{MCM}(R) = \{M \in \mathrm{mod}-R \mid \text{depth}_R(M) = \text{kr.}\dim(R)\}.$$

2. Singularity categories

Definition

Let R be two-sided Noetherian. R is called **(Iwanaga–)Gorenstein** ring if

$$\operatorname{inj.dim}_R R < \infty \text{ and } \operatorname{inj.dim} R_R < \infty.$$

The category of **maximal Cohen–Macaulay** modules (MCM) is defined as

$$\mathrm{MCM}(R) = \{M \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(M, R) = 0 \text{ for all } i > 0\} \subseteq \mathrm{mod}-R.$$

Example

(i) **Regular** rings R are Gorenstein and $\mathrm{MCM}(R) = \mathrm{proj}-R$.

(ii) **Self-injective** algebras are Gorenstein and $\mathrm{MCM}(R) = \mathrm{mod}-R$.

(iii) For **local commutative** Gorenstein rings R,

$$\mathrm{MCM}(R) = \{M \in \mathrm{mod}-R \,|\, \operatorname{depth}_R(M) = \operatorname{kr.dim}(R)\}.$$

(iv) **Hypersurface** rings $R = k[\![x_0, \ldots, x_d]\!]/(f)$ are Gorenstein.

2. Singularity categories

Remark

(i) $\mathrm{MCM}(R)$ is a **Frobenius category** with $\operatorname{proj} \mathrm{MCM}(R) = \operatorname{proj} -R$.

2. Singularity categories

Remark

(i) $\mathrm{MCM}(R)$ is a **Frobenius category** with $\operatorname{proj} \mathrm{MCM}(R) = \operatorname{proj} -R$.

(ii) In particular, $\underline{\mathrm{MCM}}(R) := \mathrm{MCM}(R)/\operatorname{proj} -R$ is **triangulated** by work of Happel.

2. Singularity categories

Remark

(i) $\mathrm{MCM}(R)$ is a **Frobenius category** with $\operatorname{proj} \mathrm{MCM}(R) = \operatorname{proj} -R$.

(ii) In particular, $\underline{\mathrm{MCM}}(R) := \mathrm{MCM}(R) / \operatorname{proj} -R$ is **triangulated** by work of Happel.

(iii) If R is **regular**, then $\underline{\mathrm{MCM}}(R) = 0$.

2. Singularity categories

Remark

(i) $\mathrm{MCM}(R)$ is a **Frobenius category** with $\operatorname{proj}\mathrm{MCM}(R) = \operatorname{proj} -R$.

(ii) In particular, $\underline{\mathrm{MCM}}(R) := \mathrm{MCM}(R)/\operatorname{proj} -R$ is **triangulated** by work of Happel.

(iii) If R is **regular**, then $\underline{\mathrm{MCM}}(R) = 0$.

Theorem (Buchweitz)

The composition of functors

$$\mathrm{MCM}(R) \subseteq \operatorname{mod} -R \subseteq \mathcal{D}^b(\operatorname{mod} -R) \to \mathcal{D}_{sg}(R)$$

2. Singularity categories

Remark

(i) $\mathrm{MCM}(R)$ is a **Frobenius category** with $\operatorname{proj}\mathrm{MCM}(R) = \operatorname{proj} - R$.

(ii) In particular, $\underline{\mathrm{MCM}}(R) := \mathrm{MCM}(R)/\operatorname{proj} - R$ is **triangulated** by work of Happel.

(iii) If R is **regular**, then $\underline{\mathrm{MCM}}(R) = 0$.

Theorem (Buchweitz)

The composition of functors

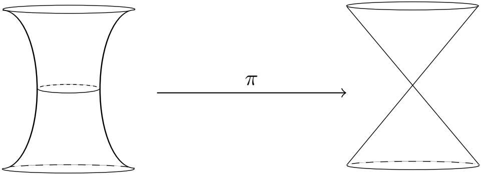
$$\mathrm{MCM}(R) \subseteq \operatorname{mod} - R \subseteq \mathcal{D}^b(\operatorname{mod} - R) \rightarrow \mathcal{D}_{sg}(R)$$

induces an **equivalence of triangulated categories**

$$\underline{\mathrm{MCM}}(R) \cong \mathcal{D}_{sg}(R).$$

Overview

3 Resolutions   1 Singularities



$$\mathcal{D}^b(\mathrm{Coh}\,Y) \xleftarrow[\text{`categorical resolution'}]{\mathbb{L}\pi^*} \mathrm{Perf}(X) \underset{\text{`smooth'}}{\subseteq} \mathcal{D}^b(\mathrm{Coh}\,X)$$

Aim

Use representation theory to describe the triangulated quotients

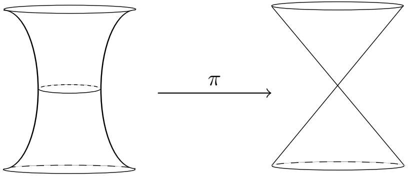
5 *Relations*

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathrm{Coh}\,Y)}{\mathrm{Perf}(X)} \longleftrightarrow \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathrm{Coh}\,X)}{\mathrm{Perf}(X)}$$

4 *Relative singularity categories*   2 *Singularity categories*

3. Resolutions

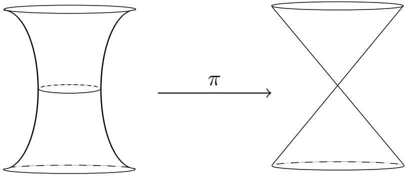
$$X = \mathsf{Spec}(R')$$



$$\mathcal{D}^b(\mathsf{Coh}\, Y) \xhookleftarrow{\mathbb{L}\pi^*} \mathsf{Perf}(X)$$

3. Resolutions

$X = \mathsf{Spec}(R')$



$$\mathcal{D}^b(A) \xleftarrow[\text{Tilting}]{\sim} \mathcal{D}^b(\mathsf{Coh}\,Y) \overset{\mathbb{L}\pi^*}{\hookleftarrow} \mathsf{Perf}(X)$$

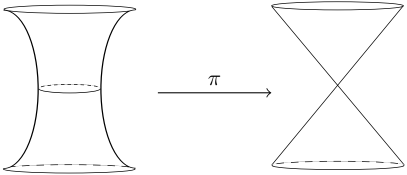
Kapranov & Vasserot

Derived McKay Correspondence

3. Resolutions

$X = \mathsf{Spec}(R')$

Preprojective algebra $\Pi\left(\, 1 \rightrightarrows 2 \,\right)$



$$\mathcal{D}^b(A) \xleftarrow[\text{Tilting}]{\sim} \mathcal{D}^b(\mathsf{Coh}\,Y) \xhookleftarrow{\mathbb{L}\pi^*} \mathsf{Perf}(X)$$

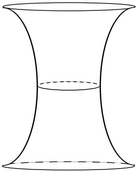
Kapranov & Vasserot

Derived McKay Correspondence

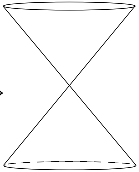
3. Resolutions

$X = \mathsf{Spec}(R')$

$$\left\{\begin{array}{ll} \text{Preprojective algebra} & \Pi\left(1 \rightrightarrows 2\right) \\ & \wr \ \text{Morita} \\ \text{Skew group algebra} & \mathbb{C}[X,Y] * \mathbb{Z}/2\mathbb{Z} \end{array}\right\}$$



$\xrightarrow{\pi}$



$$\mathcal{D}^b(A) \xleftarrow[\text{Tilting}]{\sim} \mathcal{D}^b(\mathsf{Coh}\,Y) \xhookleftarrow{\mathbb{L}\pi^*} \mathsf{Perf}(X)$$

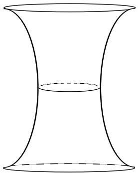
Kapranov & Vasserot

Derived McKay Correspondence

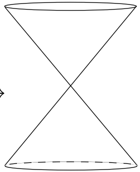
3. Resolutions

$X = \mathsf{Spec}(R')$

$$\left\{\begin{array}{cc} \text{Preprojective algebra} & \Pi\left(1 \rightrightarrows 2\right) \\ & \wr \;\text{Morita} \\ \text{Skew group algebra} & \mathbb{C}[X,Y] * \mathbb{Z}/2\mathbb{Z} \end{array}\right\}$$



$\xrightarrow{\pi}$



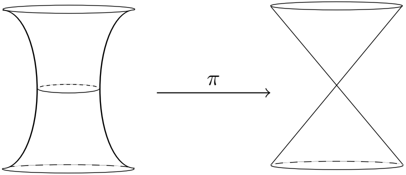
$$\mathcal{D}^b(A) \xleftarrow[\text{Tilting}]{\sim} \mathcal{D}^b(\mathsf{Coh}\,Y) \xhookleftarrow{\mathbb{L}\pi^*} \mathsf{Perf}(X)$$

$$A := \mathrm{End}_Y(\mathcal{T}) \cong \mathrm{End}_{R'}(\pi_*(\mathcal{O}_Y \oplus \mathcal{L})) \cong \mathrm{End}_{R'}(R' \oplus M)$$

3. Resolutions

$X = \mathsf{Spec}(R')$

$$\left\{\begin{array}{ll} \text{Preprojective algebra} & \Pi\left(\,1 \rightrightarrows 2\,\right) \\ & \quad \wr \ \text{Morita} \\ \text{Skew group algebra} & \mathbb{C}[X,Y] * \mathbb{Z}/2\mathbb{Z} \end{array}\right\}$$



$$\mathcal{D}^b(A) \xleftarrow[\text{Tilting}]{\sim} \mathcal{D}^b(\mathsf{Coh}\,Y) \xhookleftarrow{\mathbb{L}\pi^*} \mathsf{Perf}(X)$$

$$A := \mathrm{End}_Y(\mathcal{T}) \cong \mathrm{End}_{R'}(\pi_*(\mathcal{O}_Y \oplus \mathcal{L})) \cong \mathrm{End}_{R'}(R' \oplus M)$$

with $M \in \mathrm{MCM}(R')$.

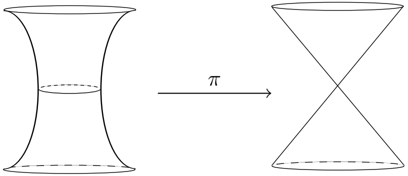
3. Resolutions

$X = \mathsf{Spec}(R')$

Preprojective algebra $\quad \Pi\left(\, 1 \rightrightarrows 2 \,\right)$

$\wr$ Morita

Skew group algebra $\quad \mathbb{C}[X,Y] * \mathbb{Z}/2\mathbb{Z}$



$$\mathcal{D}^b(A) \xleftarrow[\text{Tilting}]{\sim} \mathcal{D}^b(\mathsf{Coh}\, Y) \xhookleftarrow{\mathbb{L}\pi^*} \mathsf{Perf}(X)$$

$$A := \mathrm{End}_Y(\mathcal{T}) \cong \mathrm{End}_{R'}(\pi_*(\mathcal{O}_Y \oplus \mathcal{L})) \cong \mathrm{End}_{R'}(R' \oplus M)$$

with $M \in \mathrm{MCM}(R')$.

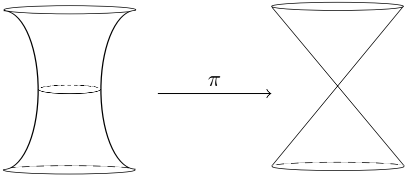
Idea (Van den Bergh)

Replace $\mathcal{D}^b(Y)$ by $\mathcal{D}^b(A)$ for a 'nice' algebra A (e.g. $\mathsf{gl.dim}(A) < \infty$)

3. Resolutions

$X = \mathsf{Spec}(R')$

$$\left\{\begin{array}{ll} \text{Preprojective algebra} & \Pi\left(\, 1 \rightrightarrows 2 \,\right) \\ & \quad\quad \wr \ \text{Morita} \\ \text{Skew group algebra} & \mathbb{C}[X, Y] * \mathbb{Z}/2\mathbb{Z} \end{array}\right\}$$



$$\mathcal{D}^b(A) \xleftarrow[\text{Tilting}]{\sim} \mathcal{D}^b(\mathsf{Coh}\, Y) \overset{\mathbb{L}\pi^*}{\hookleftarrow} \mathsf{Perf}(X)$$

$$A := \operatorname{End}_Y(\mathcal{T}) \cong \operatorname{End}_{R'}(\pi_*(\mathcal{O}_Y \oplus \mathcal{L})) \cong \operatorname{End}_{R'}(R' \oplus M)$$

with $M \in \operatorname{MCM}(R')$.

Idea (Van den Bergh)

Replace $\mathcal{D}^b(Y)$ by $\mathcal{D}^b(A)$ for a 'nice' algebra A (e.g. $\mathrm{gl.dim}(A) < \infty$) and consider it as **categorical resolution** *of X if there is an embedding*

$$\mathsf{Perf}(X) \hookrightarrow \mathcal{D}^b(A).$$

3. Resolutions

Idea (Van den Bergh)

Replace $\mathcal{D}^b(Y)$ *by* $\mathcal{D}^b(A)$ *for a 'nice' algebra* A *(e.g.* $\mathrm{gl.dim}(A) < \infty$*) and consider it as* **categorical resolution** *of* X *if there is an embedding*

$$\mathsf{Perf}(X) \hookrightarrow \mathcal{D}^b(A).$$

3. Resolutions

Idea (Van den Bergh)

Replace $\mathcal{D}^b(Y)$ by $\mathcal{D}^b(A)$ for a 'nice' algebra A (e.g. $\mathrm{gl.dim}(A) < \infty$) and consider it as **categorical resolution** *of X if there is an embedding*

$$\mathsf{Perf}(X) \hookrightarrow \mathcal{D}^b(A).$$

Definition (see Dao, Iyama, Takahashi & Vial)

Let R be an Iwanaga–Gorenstein ring. Let $M \in \mathrm{MCM}(R) := \{ N \in \mathrm{mod} - R \mid \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \}$ be a maximal Cohen–Macaulay module and $A := \mathrm{End}_R(R \oplus M)$.

3. Resolutions

Idea (Van den Bergh)

Replace $\mathcal{D}^b(Y)$ by $\mathcal{D}^b(A)$ for a 'nice' algebra A (e.g. $\operatorname{gl.dim}(A) < \infty$*) and consider it as* **categorical resolution** *of X if there is an embedding*

$$\mathsf{Perf}(X) \hookrightarrow \mathcal{D}^b(A).$$

Definition (see Dao, Iyama, Takahashi & Vial)

Let R be an Iwanaga–Gorenstein ring. Let $M \in \mathrm{MCM}(R) := \left\{ N \in \mathrm{mod} - R \,|\, \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \right\}$ be a maximal Cohen–Macaulay module and $A := \mathrm{End}_R(R \oplus M)$. If $\operatorname{gl.dim}(A) < \infty$, then A is a **non-commutative resolution** (NCR) and $\mathcal{D}^b(A)$ is a **categorical resolution** of R.

3. Resolutions

Definition (see Dao, Iyama, Takahashi & Vial)

Let R be an Iwanaga–Gorenstein ring. Let $M \in \mathrm{MCM}(R) := \{ N \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \}$ be a maximal Cohen–Macaulay module and $A := \mathrm{End}_R(R \oplus M)$. If $\mathrm{gl.dim}(A) < \infty$, then A is a **non-commutative resolution** (NCR) and $\mathcal{D}^b(A)$ is a **categorical resolution** of R.

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k)$$

3. Resolutions

Definition (see Dao, Iyama, Takahashi & Vial)

Let R be an Iwanaga–Gorenstein ring. Let $M \in \mathrm{MCM}(R) := \{ N \in \mathrm{mod}-R \mid \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \}$ be a maximal Cohen–Macaulay module and $A := \mathrm{End}_R(R \oplus M)$. If gl. dim$(A) < \infty$, then A is a **non-commutative resolution** (NCR) and $\mathcal{D}^b(A)$ is a **categorical resolution** of R.

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathrm{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\big(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \big)/(ab)$$

3. Resolutions

Definition (see Dao, Iyama, Takahashi & Vial)

Let R be an Iwanaga–Gorenstein ring. Let $M \in \mathrm{MCM}(R) := \left\{ N \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \right\}$ be a maximal Cohen–Macaulay module and $A := \mathrm{End}_R(R \oplus M)$. If gl. dim$(A) < \infty$, then A is a **non-commutative resolution** (NCR) and $\mathcal{D}^b(A)$ is a **categorical resolution** of R.

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$$

has global dimension 2.

3. Resolutions

Definition (see Dao, Iyama, Takahashi & Vial)

Let R be an Iwanaga–Gorenstein ring. Let $M \in \mathrm{MCM}(R) := \{ N \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(N,R) = 0 \text{ for all } i > 0 \}$ be a maximal Cohen–Macaulay module and $A := \mathrm{End}_R(R \oplus M)$. If $\mathrm{gl.dim}(A) < \infty$, then A is a **non-commutative resolution** (NCR) and $\mathcal{D}^b(A)$ is a **categorical resolution** of R.

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$$

has global dimension 2. In particular, $\mathsf{Aus}(R)$ is an NCR of R.

3. Resolutions

Definition (see Dao, Iyama, Takahashi & Vial)

Let R be an Iwanaga–Gorenstein ring. Let $M \in \mathrm{MCM}(R) := \left\{ N \in \mathrm{mod} - R \,|\, \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \right\}$ be a maximal Cohen–Macaulay module and $A := \mathrm{End}_R(R \oplus M)$. If $\mathrm{gl.dim}(A) < \infty$, then A is a **non-commutative resolution** (NCR) and $\mathcal{D}^b(A)$ is a **categorical resolution** of R.

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$$

has global dimension 2. In particular, $\mathsf{Aus}(R)$ is an NCR of R.

- For $k = \mathbb{C}$ there is an equivalence of categories

$$\mathrm{mod} - \mathsf{Aus}(R) \cong \mathcal{O}_0\big(\mathfrak{sl}_2(\mathbb{C})\big) \quad (\textbf{principal block of category } \mathcal{O})$$

3. Resolutions

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$$

has global dimension 2. In particular, $\mathsf{Aus}(R)$ is an **NCR** of R.

3. Resolutions

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\, 1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \,\Big)/(ab)$$

has global dimension 2. In particular, $\mathsf{Aus}(R)$ is an **NCR** of R.

- For $k = \mathbb{C}$ there is an **equivalence** of categories

$$\mathrm{mod} - \mathsf{Aus}(R) \cong \mathcal{O}_0\big(\mathfrak{sl}_2(\mathbb{C})\big) \quad \text{(principal block of category } \mathcal{O}\text{)}$$

3. Resolutions

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \big)/(ab)$$

has global dimension 2. In particular, $\mathsf{Aus}(R)$ is an **NCR** of R.

- For $k = \mathbb{C}$ there is an **equivalence** of categories

$$\mathrm{mod} - \mathsf{Aus}(R) \cong \mathcal{O}_0\big(\mathfrak{sl}_2(\mathbb{C})\big) \quad \text{(principal block of category } \mathcal{O}\text{)}$$

- If $\mathrm{char}\, k = 2$, then the **Schur algebra** $S_k(2,2)$ is Morita-equivalent to $\mathsf{Aus}(R)$. Moreover, it is an NCR of the group algebra $k[\mathbb{Z}_2] \cong R$.

3. Resolutions

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$$

has global dimension 2. In particular, $\mathsf{Aus}(R)$ is an **NCR** of R.

- For $k = \mathbb{C}$ there is an **equivalence** of categories

$$\mathrm{mod} - \mathsf{Aus}(R) \cong \mathcal{O}_0\big(\mathfrak{sl}_2(\mathbb{C})\big) \quad \text{(principal block of category } \mathcal{O}\text{)}$$

- If $\mathrm{char}\, k = 2$, then the **Schur algebra** $S_k(2,2)$ is Morita-equivalent to $\mathsf{Aus}(R)$. Moreover, it is an NCR of the group algebra $k[\mathbb{Z}_2] \cong R$.
- More generally, if $\mathrm{char}\, k = p$, then the Schur algebra $S_k(p,p)$ gives an NCR of the group algebra $k[\mathfrak{S}_p]$.

3. Resolutions

Theorem (Auslander, Auslander & Roggenkamp, Iyama)

Let R *be a commutative complete Gorenstein* k*-algebra of Krull dimension* d*, with* $k \cong R/\mathfrak{m}$ *algebraically closed.*

3. Resolutions

Theorem (Auslander, Auslander & Roggenkamp, Iyama)

Let R be a commutative complete Gorenstein k-algebra of Krull dimension d, with $k \cong R/\mathfrak{m}$ algebraically closed. If R has only finitely many indecomposable MCMs, then the **Auslander algebra**

$$\mathsf{Aus}(R) := \mathrm{End}_R\left(\bigoplus_{M \in \mathrm{ind}\, \mathrm{MCM}(R)} M\right)$$

3. Resolutions

Theorem (Auslander, Auslander & Roggenkamp, Iyama)

Let R be a commutative complete Gorenstein k-algebra of Krull dimension d, with $k \cong R/\mathfrak{m}$ algebraically closed. If R has only finitely many indecomposable MCMs, then the **Auslander algebra**

$$\mathsf{Aus}(R) := \mathrm{End}_R\left(\bigoplus_{M \in \mathrm{ind}\, \mathrm{MCM}(R)} M\right)$$

has global dimension **at most** $\max\{2, \mathsf{kr.\,dim}(R)\}$.

3. Resolutions

Theorem (Auslander, Auslander & Roggenkamp, Iyama)

Let R be a commutative complete Gorenstein k-algebra of Krull dimension d, with $k \cong R/\mathfrak{m}$ algebraically closed. If R has only finitely many indecomposable MCMs, then the **Auslander algebra**

$$\mathsf{Aus}(R) := \mathrm{End}_R \left(\bigoplus_{M \in \mathrm{ind}\, \mathrm{MCM}(R)} M \right)$$

has global dimension **at most** $\max\{2, \mathsf{kr.dim}(R)\}$.
In particular, $\mathsf{Aus}(R)$ *is an* **NCR** *of* R.

3. Resolutions

Theorem (Auslander, Auslander & Roggenkamp, Iyama)

Let R be a commutative complete Gorenstein k-algebra of Krull dimension d, with $k \cong R/\mathfrak{m}$ algebraically closed. If R has only finitely many indecomposable MCMs, then the **Auslander algebra**

$$\mathsf{Aus}(R) := \mathrm{End}_R\left(\bigoplus_{M \in \mathrm{ind}\,\mathrm{MCM}(R)} M\right)$$

has global dimension **at most** $\max\{2, \mathsf{kr.\,dim}(R)\}$.
In particular, $\mathsf{Aus}(R)$ *is an* **NCR** *of* R.

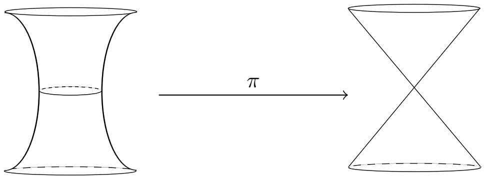
Remark

All ADE-singularities satisfy the conditions of this theorem, e.g. type (A_n):

$$k[\![x_0, \ldots, x_d]\!]/(x_0^{n+1} + x_1^2 + \ldots + x_d^2).$$

Overview

3 Resolutions 1 Singularities



$$\mathcal{D}^b(\mathsf{Coh}\, Y) \xleftarrow[\text{`categorical resolution'}]{\mathbb{L}\pi^*} \mathsf{Perf}(X) \underset{\text{`smooth'}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\, X)$$

Aim

Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathsf{Coh}\, Y)}{\mathsf{Perf}(X)} \xleftrightarrow{\text{⑤ Relations}} \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathsf{Coh}\, X)}{\mathsf{Perf}(X)}$$

4 *Relative singularity categories* 2 *Singularity categories*

4. Relative singularity categories

Let A be an NCR of R, i.e.

$$A = \mathrm{End}_R(R \oplus M) \cong \begin{pmatrix} R & M \\ \mathrm{Hom}_R(R, M) & \mathrm{End}_R(M) \end{pmatrix}$$

4. Relative singularity categories

Let A be an NCR of R, i.e.

$$A = \mathrm{End}_R(R \oplus M) \cong \begin{pmatrix} R & M \\ \mathrm{Hom}_R(R, M) & \mathrm{End}_R(M) \end{pmatrix} \ni \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: e$$

4. Relative singularity categories

Let A be an NCR of R, i.e.

$$A = \mathrm{End}_R(R \oplus M) \cong \begin{pmatrix} R & M \\ \mathrm{Hom}_R(R, M) & \mathrm{End}_R(M) \end{pmatrix} \ni \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: e$$

Then there is an **embedding of triangulated categories**

$$\begin{array}{ccccc} K^b(\mathrm{proj} - R) & \xrightarrow{=} & K^b(\mathrm{proj} - eAe) & \xhookrightarrow{- \otimes_{eAe} eA} & \mathcal{D}^b(A) \\ R & \longmapsto & eAe & \longmapsto & eA \end{array}$$

4. Relative singularity categories

Let A be an NCR of R, i.e.

$$A = \mathrm{End}_R(R \oplus M) \cong \begin{pmatrix} R & M \\ \mathrm{Hom}_R(R, M) & \mathrm{End}_R(M) \end{pmatrix} \ni \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: e$$

Then there is an **embedding of triangulated categories**

$$\begin{array}{ccccc} K^b(\mathrm{proj} - R) & \xrightarrow{=} & K^b(\mathrm{proj} - eAe) & \xhookrightarrow{- \otimes_{eAe} eA} & \mathcal{D}^b(A) \\ R & \longmapsto & eAe & \longmapsto & eA \end{array}$$

Definition

The **relative singularity category** of the NCR A of R is the triangulated quotient category

$$\Delta_R(A) := \frac{\mathcal{D}^b(A)}{K^b(\mathrm{proj} - R)} = \frac{\mathcal{D}^b(A)}{\mathrm{thick}(eA)}$$

4. Relative singularity categories

Definition

The **relative singularity category** of the NCR A of R is the triangulated quotient category

$$\Delta_R(A) := \frac{\mathcal{D}^b(A)}{K^b(\mathrm{proj}-R)} = \frac{\mathcal{D}^b(A)}{\mathsf{thick}(eA)}$$

Remark

$\Delta_R(A)$ may be viewed as a **measure** for the **size** of the **resolution** $\mathcal{D}^b(A)$ relative to the 'smooth part' $K^b(\mathrm{proj}-R) \subseteq \mathcal{D}^b(A)$.

4. Relative singularity categories

Definition

The **relative singularity category** of the NCR A of R is the triangulated quotient category

$$\Delta_R(A) := \frac{\mathcal{D}^b(A)}{K^b(\mathrm{proj} - R)} = \frac{\mathcal{D}^b(A)}{\mathsf{thick}(eA)}$$

Remark

$\Delta_R(A)$ may be viewed as a **measure** for the **size** of the **resolution** $\mathcal{D}^b(A)$ relative to the 'smooth part' $K^b(\mathrm{proj} - R) \subseteq \mathcal{D}^b(A)$.

Remark

Relative singularity categories were also studied by X.-W. Chen and Thanhoffer de Völcsey & Van den Bergh.

4. Relative singularity categories

Example

Let $R = k[x]/(x^2)$ and $A = \mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$ be the **Auslander algebra** of R.

Example

Let $R = k[x]/(x^2)$ and $A = \mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$ be the **Auslander algebra** of R. Then $e = e_1$, so R corresponds to P_1.

Example

Let $R = k[x]/(x^2)$ and $A = \mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \big)/(ab)$ be the **Auslander algebra** of R. Then $e = e_1$, so R corresponds to P_1. The indecomposable objects in $\mathcal{D}^b(A)$ are given by

Example

Let $R = k[x]/(x^2)$ and $A = \mathsf{Aus}(R) = \operatorname{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$ be the **Auslander algebra** of R. Then $e = e_1$, so R corresponds to P_1. The indecomposable objects in $\mathcal{D}^b(A)$ are given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \overbrace{P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1}^{\mathbb{N}_0 \ni l \text{ terms}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \qquad \in K^b(\operatorname{proj} - R)$$

Example

Let $R = k[x]/(x^2)$ and $A = \mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \big)/(ab)$ be the **Auslander algebra** of R. Then $e = e_1$, so R corresponds to P_1. The indecomposable objects in $\mathcal{D}^b(A)$ are given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \overbrace{P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1}^{\mathbb{N}_0 \ni l \text{ terms}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \qquad \in K^b(\mathrm{proj} - R)$$

$$\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Example

Let $R = k[x]/(x^2)$ and $A = \mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \Big)/(ab)$ be the **Auslander algebra** of R. Then $e = e_1$, so R corresponds to P_1. The indecomposable objects in $\mathcal{D}^b(A)$ are given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \overbrace{P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1}^{\mathbb{N}_0 \ni l \text{ terms}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \qquad \in K^b(\mathrm{proj}-R)$$

$$\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots$$

Example

Let $R = k[x]/(x^2)$ and $A = \mathrm{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\big(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \big)/(ab)$ be the **Auslander algebra** of R. Then $e = e_1$, so R corresponds to P_1. The indecomposable objects in $\mathcal{D}^b(A)$ are given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \overbrace{P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1}^{\mathbb{N}_0 \ni l \text{ terms}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \qquad \in K^b(\mathrm{proj}-R)$$

$$\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots$$

Example

Let $R = k[x]/(x^2)$ and $A = \mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \right)/(ab)$ be the **Auslander algebra** of R. Then $e = e_1$, so R corresponds to P_1. The indecomposable objects in $\mathcal{D}^b(A)$ are given by

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \overbrace{P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1}^{\mathbb{N}_0 \ni l \text{ terms}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \qquad \in K^b(\mathrm{proj} - R)$$

$$\left.\begin{array}{l}
\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \\
\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots \\
\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots
\end{array}\right\} \in \mathcal{D}^b(A)$$

$$K^b(\mathrm{proj} - R) \hookrightarrow \mathcal{D}^b(A)$$

4. Relative singularity categories

Example

$$\begin{array}{ll}
\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots & \in K^b(\mathrm{proj}-R) \\
\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots & \big\downarrow \\
\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots & \Bigg\} \in \mathcal{D}^b(A) \\
\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots &
\end{array}$$

Now, $K^b(\mathrm{proj}-R)$ vanishes in the relative singularity category $\Delta_R(A)$

Example

$$\left.\begin{array}{l}\cdots\longrightarrow 0\longrightarrow 0\longrightarrow P_1\xrightarrow{ba\cdot}P_1\xrightarrow{ba\cdot}\cdots\xrightarrow{ba\cdot}P_1\longrightarrow 0\longrightarrow 0\longrightarrow\cdots \qquad \in K^b(\operatorname{proj}-R)\end{array}\right.$$

$$\left.\begin{array}{l}\cdots\longrightarrow 0\longrightarrow P_2\xrightarrow{b\cdot}P_1\xrightarrow{ba\cdot}P_1\xrightarrow{ba\cdot}\cdots\xrightarrow{ba\cdot}P_1\longrightarrow 0\longrightarrow 0\longrightarrow\cdots\\ \cdots\longrightarrow 0\longrightarrow 0\longrightarrow P_1\xrightarrow{ba\cdot}P_1\xrightarrow{ba\cdot}\cdots\xrightarrow{ba\cdot}P_1\xrightarrow{a\cdot}P_2\longrightarrow 0\longrightarrow\cdots\\ \cdots\longrightarrow 0\longrightarrow P_2\xrightarrow{b\cdot}P_1\xrightarrow{ba\cdot}P_1\xrightarrow{ba\cdot}\cdots\xrightarrow{ba\cdot}P_1\xrightarrow{a\cdot}P_2\longrightarrow 0\longrightarrow\cdots\end{array}\right\}\in\mathcal{D}^b(A)$$

$$K^b(\operatorname{proj}-R)\hookrightarrow\mathcal{D}^b(A)$$

Now, $K^b(\operatorname{proj}-R)$ vanishes in the relative singularity category $\Delta_R(A)$ and as in the computations for singularity categories, the morphism

$$\begin{array}{ccccccccccccccc} X = & \cdots\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \overbrace{P_1\xrightarrow{ba\cdot}P_1\xrightarrow{ba\cdot}\cdots\xrightarrow{ba\cdot}P_1}^{\mathbb{N}\ni l\text{ terms}} & \longrightarrow & 0 & \longrightarrow\cdots & \in K^b(\operatorname{proj}-R)\\ \big\downarrow f & & \big\downarrow & & \big\downarrow & & \big\downarrow\mathrm{id}\ \ \big\downarrow\mathrm{id}\ \ \ \big\downarrow\mathrm{id} & & \big\downarrow & & \\ Y = & \cdots\longrightarrow & 0 & \longrightarrow & P_2 & \xrightarrow{b\cdot} & P_1\xrightarrow{ba\cdot}P_1\xrightarrow{ba\cdot}\cdots\xrightarrow{ba\cdot}P_1 & \longrightarrow & 0 & \longrightarrow\cdots & \end{array}$$

4. Relative singularity categories

Example

...as in the computations for singularity categories, the morphism

$$\begin{tikzcd}
 & & & & \overbrace{\hphantom{P_1 \to P_1 \to \cdots \to P_1}}^{\mathbb{N} \ni l \text{ terms}} & & & & \\
X = \cdots \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d] & \cdots \qquad \in K^b(\mathrm{proj} -R) \\
Y = \cdots \arrow[r] & 0 \arrow[r] & P_2 \arrow[r, "b\cdot"] & P_1 \arrow[r, "ba\cdot"] & P_1 \arrow[r, "ba\cdot"] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

shows that there exists $n \in \mathbb{Z}$ such that

$$Y \cong \Sigma^n(P_2) \qquad \text{in } \Delta_R(A).$$

4. Relative singularity categories

Example

...as in the computations for singularity categories, the morphism

$$\begin{tikzcd}[column sep=small]
X = \cdots \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d] & \cdots \\
Y = \cdots \arrow[r] & 0 \arrow[r] & P_2 \arrow[r, "b\cdot"] & P_1 \arrow[r, "ba\cdot"] & P_1 \arrow[r, "ba\cdot"] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd} \quad \in K^b(\mathrm{proj} - R)$$

where the top row has $\overbrace{P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1}^{\mathbb{N} \ni l \text{ terms}}$

shows that there exists $n \in \mathbb{Z}$ such that

$$Y \cong \Sigma^n(P_2) \qquad \text{in } \Delta_R(A).$$

Similarly, one obtains isomorphisms

$$\cdots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots \cong \Sigma^n(P_2).$$

4. Relative singularity categories

Example

One can show that the remaining objects, i.e. shifts of

$$P_2 \quad \text{and} \quad \cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots$$

are indecomposable and pairwise non-isomorphic in $\Delta_R(A)$.

Example

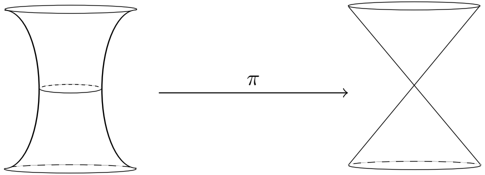
One can show that the remaining objects, i.e. shifts of

$$P_2 \quad \text{and} \quad \cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots$$

are indecomposable and pairwise non-isomorphic in $\Delta_R(A)$. Moreover, the **quiver of irreducible morphisms** of $\Delta_R(A)$ consists of one $\mathbb{Z}A_\infty$-component and one equioriented A_∞^∞-component.

Overview

3 Resolutions 1 Singularities



$$\mathcal{D}^b(\mathsf{Coh}\, Y) \xleftarrow[\text{'categorical resolution'}]{\mathbb{L}\pi^*} \mathsf{Perf}(X) \underset{\text{'smooth'}}{\subseteq} \mathcal{D}^b(\mathsf{Coh}\, X)$$

Aim

Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathsf{Coh}\, Y)}{\mathsf{Perf}(X)} \xleftrightarrow{\text{5 } \textit{Relations}} \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathsf{Coh}\, X)}{\mathsf{Perf}(X)}$$

4 *Relative singularity categories* 2 *Singularity categories*

5. Relations: Knörrer's Periodicity

Knörrer's Periodicity Theorem yields a relation between singularity categories for different Krull dimensions:

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+x^2+y^2)\big),$$

where $S = k[\![z_0, \ldots, z_d]\!]$, $f \in (z_0, \ldots, z_d)$ and $d \geq 0$.

5. Relations: Knörrer's Periodicity

Knörrer's Periodicity Theorem yields a relation between singularity categories for different Krull dimensions:

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+x^2+y^2)\big),$$

where $S = k[\![z_0, \ldots, z_d]\!]$, $f \in (z_0, \ldots, z_d)$ and $d \geq 0$.

Example

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$,

5. Relations: Knörrer's Periodicity

Knörrer's Periodicity Theorem yields a relation between singularity categories for different Krull dimensions:

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+x^2+y^2)\big),$$

where $S = k[\![z_0,\dots,z_d]\!]$, $f \in (z_0,\dots,z_d)$ and $d \geq 0$.

Example

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$, then

$$\mathcal{D}_{sg}(R) \xrightarrow[\sim]{\text{Knörrer}} \mathcal{D}_{sg}(R')$$

5. Relations: Knörrer's Periodicity

Knörrer's Periodicity Theorem yields a relation between singularity categories for different Krull dimensions:

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+x^2+y^2)\big),$$

where $S = k[\![z_0,\ldots,z_d]\!]$, $f \in (z_0,\ldots,z_d)$ and $d \geq 0$.

Example

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$, then

$$\mathcal{D}_{sg}(R) \xrightarrow[\sim]{\text{Knörrer}} \mathcal{D}_{sg}(R') \xleftarrow[\sim]{\Gamma(-)} \mathcal{D}_{sg}\Big($$  $$\Big)$$

Knörrer's Periodicity Theorem yields a relation between singularity categories for different Krull dimensions:

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+x^2+y^2)\big),$$

where $S = k[\![z_0,\ldots,z_d]\!]$, $f \in (z_0,\ldots,z_d)$ and $d \geq 0$.

Example

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$, then

$$\operatorname{mod}-k \cong \mathcal{D}_{sg}(R) \xrightarrow[\sim]{\text{Knörrer}} \mathcal{D}_{sg}(R') \xleftarrow[\sim]{\Gamma(-)} \mathcal{D}_{sg}\Big($$  $$\Big)$$

5. Relations: Knörrer's Periodicity

Knörrer's Periodicity Theorem yields a relation between singularity categories for different Krull dimensions:

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+x^2+y^2)\big),$$

where $S = k[\![z_0,\ldots,z_d]\!]$, $f \in (z_0,\ldots,z_d)$ and $d \geq 0$.

Example

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$, then

$\operatorname{mod} -k \cong \mathcal{D}_{sg}(R) \xrightarrow[\sim]{\text{Knörrer}} \mathcal{D}_{sg}(R') \xleftarrow[\sim]{\Gamma(-)} \mathcal{D}_{sg}\Big($  $\Big)$

$\Rightarrow$ Get representation-theoretic description of $\mathcal{D}_{sg}\Big($  $\Big)$

5. Relations: Knörrer's Periodicity

Knörrer's Periodicity Theorem yields a relation between singularity categories for different Krull dimensions:

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+x^2+y^2)\big),$$

where $S = k[\![z_0,\ldots,z_d]\!]$, $f \in (z_0,\ldots,z_d)$ and $d \geq 0$.

Example

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$, then

$$\mathrm{mod}-k \cong \mathcal{D}_{sg}(R) \xrightarrow[\sim]{\text{Knörrer}} \mathcal{D}_{sg}(R') \xleftarrow[\sim]{\Gamma(-)} \mathcal{D}_{sg}\Big($$  $$\Big)$$

$\Rightarrow$ Get representation-theoretic description of $\mathcal{D}_{sg}\Big($  $\Big)$

Question

Does a similar result hold for **relative singularity categories***?*

5. Relations: Main result

Question

Does a similar result hold for **relative singularity categories***?*

The following result gives a first answer to this question:

5. Relations: Main result

Question

Does a similar result hold for **relative singularity categories***?*

The following result gives a first answer to this question:

Theorem (K.-Yang)

Let R and R' be MCM–**representation finite** *complete Gorenstein k-algebras with* **Auslander algebras** $A = \mathsf{Aus}(R)$ *respectively* $A' = \mathsf{Aus}(R')$.

5. Relations: Main result

Question

Does a similar result hold for **relative singularity categories***?*

The following result gives a first answer to this question:

Theorem (K.-Yang)

Let R and R' be MCM–**representation finite** *complete Gorenstein k-algebras with* **Auslander algebras** $A = \mathsf{Aus}(R)$ *respectively $A' = \mathsf{Aus}(R')$. Then the following statements are equivalent.*

(i) *There is a triangle equivalence $\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$.*

5. Relations: Main result

Question

Does a similar result hold for **relative singularity categories***?*

The following result gives a first answer to this question:

Theorem (K.-Yang)

Let R *and* R' *be* MCM–**representation finite** *complete Gorenstein* k*-algebras with* **Auslander algebras** $A = \mathsf{Aus}(R)$ *respectively* $A' = \mathsf{Aus}(R')$. *Then the following statements are equivalent.*

(i) *There is a triangle equivalence* $\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$.

(ii) *There is a triangle equivalence* $\Delta_R(A) \cong \Delta_{R'}(A')$.

5. Relations: Main result

Question

Does a similar result hold for **relative singularity categories***?*

The following result gives a first answer to this question:

Theorem (K.-Yang)

Let R and R' be MCM–**representation finite** *complete Gorenstein k-algebras with* **Auslander algebras** $A = \mathsf{Aus}(R)$ *respectively* $A' = \mathsf{Aus}(R')$*. Then the following statements are equivalent.*

(i) *There is a triangle equivalence* $\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$.

(ii) *There is a triangle equivalence* $\Delta_R(A) \cong \Delta_{R'}(A')$.

The implication $(ii) \Rightarrow (i)$ holds more generally for arbitrary NCRs A and A' of arbitrary isolated Gorenstein singularities R and R'.

5. Relations: Main result

Question

Does a similar result hold for **relative singularity categories***?*

The following result gives a first answer to this question:

Theorem (K.-Yang)

Let R and R' be MCM–**representation finite** *complete Gorenstein k-algebras with* **Auslander algebras** $A = \mathsf{Aus}(R)$ *respectively* $A' = \mathsf{Aus}(R')$. *Then the following statements are equivalent.*

(i) *There is a triangle equivalence* $\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$.

(ii) *There is a triangle equivalence* $\Delta_R(A) \cong \Delta_{R'}(A')$.

The implication $(ii) \Rightarrow (i)$ holds more generally for arbitrary NCRs A and A' of arbitrary isolated Gorenstein singularities R and R'. In fact, there **always exists a quotient functor** $\Delta_R(A) \to \mathcal{D}_{sg}(R)$.

5. Relations: Application

Let $R = k[x]/(x^2)$ and $R' = k[\![x, y, z]\!]/(x^2 + y^2 + z^2)$.

5. Relations: Application

Let $R = k[x]/(x^2)$ and $R' = k[\![x, y, z]\!]/(x^2 + y^2 + z^2)$.

$\mathcal{D}^b\Big($  $\Big)$

$\overline{\mathrm{Perf}\Big(}$  $\Big)$

$$\Big\downarrow \wr \quad \text{Derived McKay}$$

$$\frac{\mathcal{D}^b(\mathsf{Aus}(R'))}{K^b(\mathrm{proj}-R')}$$

5. Relations: Application

Let $R = k[x]/(x^2)$ and $R' = k[\![x, y, z]\!]/(x^2 + y^2 + z^2)$.

$\mathcal{D}^b($  $)$

$\overline{\mathrm{Perf}(}$  $)$

$$\Big\downarrow \wr \quad \text{Derived McKay}$$

$$\frac{\mathcal{D}^b(\mathsf{Aus}(R'))}{K^b(\mathrm{proj}-R')}$$

$$\Big\downarrow \wr \quad \text{K.-Yang}$$

$$\frac{\mathcal{D}^b(\mathsf{Aus}(R))}{K^b(\mathrm{proj}-R)}$$

5. Relations: Application

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$.



$$\begin{tikzcd}
\dfrac{\mathcal{D}^b(\quad)}{\mathrm{Perf}(\quad)} \arrow[r, two heads, "\mathsf{R}\pi_*"] \arrow[d, "\wr"', "\text{Derived McKay}"] & \mathcal{D}_{sg}(\quad) \arrow[d, "\wr"', "\Gamma(-)"] \\
\dfrac{\mathcal{D}^b(\mathsf{Aus}(R'))}{K^b(\mathrm{proj}-R')} \arrow[r, two heads] \arrow[d, "\wr"', "\text{K.-Yang}"] & \mathcal{D}_{sg}(R') \arrow[d, "\wr"', "\text{Knörrer}"] \\
\dfrac{\mathcal{D}^b(\mathsf{Aus}(R))}{K^b(\mathrm{proj}-R)} \arrow[r, two heads] & \mathcal{D}_{sg}(R)
\end{tikzcd}$$

5. Relations: Application

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$.

$$\begin{tikzcd}
\langle \mathcal{O}_E(-1) \rangle \arrow[r, hook] & \dfrac{\mathcal{D}^b(\quad)}{\mathrm{Perf}(\quad)} \arrow[r, two heads, "\mathrm{R}\pi_*"] \arrow[d, "\wr"', "\text{Derived McKay}"] & \mathcal{D}_{sg}(\quad) \arrow[d, "\wr"', "\Gamma(-)"] \\
 & \dfrac{\mathcal{D}^b(\mathsf{Aus}(R'))}{K^b(\mathrm{proj}-R')} \arrow[r, two heads] \arrow[d, "\wr"', "\text{K.-Yang}"] & \mathcal{D}_{sg}(R') \arrow[d, "\wr"', "\text{Knörrer}"] \\
 & \dfrac{\mathcal{D}^b(\mathsf{Aus}(R))}{K^b(\mathrm{proj}-R)} \arrow[r, two heads] & \mathcal{D}_{sg}(R)
\end{tikzcd}$$

5. Relations: Application

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$.

$$\begin{tikzcd}
\langle \mathcal{O}_E(-1) \rangle \arrow[r, hook] \arrow[d, "\wr"'] & \dfrac{\mathcal{D}^b(\quad)}{\mathrm{Perf}(\quad)} \arrow[r, two heads, "\mathsf{R}\pi_*"] \arrow[d, "\wr"', "\text{Derived McKay}"] & \mathcal{D}_{sg}(\quad) \arrow[d, "\wr"', "\Gamma(-)"] \\
\langle S_2' \rangle \arrow[r, hook] \arrow[d, "\wr"'] & \dfrac{\mathcal{D}^b(\mathsf{Aus}(R'))}{K^b(\mathrm{proj} - R')} \arrow[r, two heads] \arrow[d, "\wr"', "\text{K.-Yang}"] & \mathcal{D}_{sg}(R') \arrow[d, "\wr"', "\text{Knörrer}"] \\
\langle S_2 \rangle \arrow[r, hook] & \dfrac{\mathcal{D}^b(\mathsf{Aus}(R))}{K^b(\mathrm{proj} - R)} \arrow[r, two heads] & \mathcal{D}_{sg}(R)
\end{tikzcd}$$

5. Relations: Application

Let $R = k[x]/(x^2)$ and $R' = k[\![x,y,z]\!]/(x^2+y^2+z^2)$.

$$\begin{tikzcd}
\langle \mathcal{O}_E(-1) \rangle \arrow[r, hook] \arrow[d, "\wr"'] & \dfrac{\mathcal{D}^b(\quad)}{\mathrm{Perf}(\quad)} \arrow[r, two heads, "\mathsf{R}\pi_*"] \arrow[d, "\wr"', "\text{Derived McKay}"] & \mathcal{D}_{sg}(\quad) \arrow[d, "\wr"', "\Gamma(-)"] \\
\langle S_2' \rangle \arrow[r, hook] \arrow[d, "\wr"'] & \dfrac{\mathcal{D}^b(\mathsf{Aus}(R'))}{K^b(\mathrm{proj}-R')} \arrow[r, two heads] \arrow[d, "\wr"', "\text{K.-Yang}"] & \mathcal{D}_{sg}(R') \arrow[d, "\wr"', "\text{Knörrer}"] \\
\langle S_2 \rangle \arrow[r, hook] & \dfrac{\mathcal{D}^b(\mathsf{Aus}(R))}{K^b(\mathrm{proj}-R)} \arrow[r, two heads] & \mathcal{D}_{sg}(R)
\end{tikzcd}$$

5. Relations: Application

Let $R = k[x]/(x^2)$ and $R' = k[\![x, y, z]\!]/(x^2 + y^2 + z^2)$.

$$\begin{tikzcd}
\langle \mathcal{O}_E(-1) \rangle \arrow[r, hook] \arrow[d, "\wr"'] & \dfrac{\mathcal{D}^b(\quad)}{\mathrm{Perf}(\quad)} \arrow[r, two heads, "\mathsf{R}\pi_*"] \arrow[d, "\wr"', "\text{Derived McKay}"] & \mathcal{D}_{sg}(\quad) \arrow[d, "\wr"', "\Gamma(-)"] \\
\langle S_2' \rangle \arrow[r, hook] \arrow[d, "\wr"'] & \dfrac{\mathcal{D}^b(\mathsf{Aus}(R'))}{K^b(\mathrm{proj}-R')} \arrow[r, two heads] \arrow[d, "\wr"', "\text{K.-Yang}"] & \mathcal{D}_{sg}(R') \arrow[d, "\wr"', "\text{Knörrer}"] \\
\langle S_2 \rangle \arrow[r, hook] & \dfrac{\mathcal{D}^b(\mathsf{Aus}(R))}{K^b(\mathrm{proj}-R)} \arrow[r, two heads] & \mathcal{D}_{sg}(R)
\end{tikzcd}$$

Summary

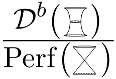
- Using '**Knörrer Periodicity**' for **relative singularity categories**, we get an **explicit description** of the quotient category

$$\frac{\mathcal{D}^b(\quad)}{\operatorname{Perf}(\quad)}$$

in terms of **representation theory** of **finite dimensional algebras.**

Summary

- Using '**Knörrer Periodicity**' for **relative singularity categories**, we get an **explicit description** of the quotient category



in terms of **representation theory** of **finite dimensional algebras.**

- Moreover, this category is an 'extension' of two **cluster categories**:
  - $\mathcal{D}_{sg}\big(k[x]/(x^2)\big)$ (1-cluster category of type $\mathbb{A}_1$).
  - $\langle S_2 \rangle$ (cluster category of type $\mathbb{A}_\infty$, see Holm & Jorgensen).

Summary

- Using '**Knörrer Periodicity**' for **relative singularity categories**, we get an **explicit description** of the quotient category

$$\frac{\mathcal{D}^b(\;)}{\mathrm{Perf}(\;)}$$

  in terms of **representation theory** of **finite dimensional algebras.**
- Moreover, this category is an 'extension' of two **cluster categories**:
  - $\mathcal{D}_{sg}\big(k[x]/(x^2)\big)$ (1-cluster category of type $\mathbb{A}_1$).
  - $\langle S_2 \rangle$ (cluster category of type $\mathbb{A}_\infty$, see Holm & Jorgensen).
- More generally for ADE-singularities R, the **relative Auslander singularity categories** $\Delta_R(\mathsf{Aus}(R))$ admit **explicit dg descriptions**.

- Using '**Knörrer Periodicity**' for **relative singularity categories**, we get an **explicit description** of the quotient category

$\mathcal{D}^b($  $)$

$\mathrm{Perf}($  $)$

in terms of **representation theory** of **finite dimensional algebras.**

- Moreover, this category is an 'extension' of two **cluster categories**:
  - $\mathcal{D}_{sg}\big(k[x]/(x^2)\big)$ (1-cluster category of type $\mathbb{A}_1$).
  - $\langle S_2 \rangle$ (cluster category of type $\mathbb{A}_\infty$, see Holm & Jorgensen).
- More generally for ADE-singularities R, the **relative Auslander singularity categories** $\Delta_R(\mathsf{Aus}(R))$ admit **explicit dg descriptions**.
- Finally, in dimension 3, the '**conifold**'

$$X = V(W^2 + X^2 + Y^2 + Z^2) \subseteq \mathbb{A}^4$$

has a crepant resolution Y and the **relative singularity category** $\Delta_X(Y)$ admits a very similar representation theoretic description.

Thank you!