Ringel-duality for certain ultra strongly quasi-hereditary algebras inspired by Knörrer-type equivalences for two-dimensional cyclic quotient singularities

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Motivation

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(ii) Ringel-dual $(A) \cong A \cong A^{op}$

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Explain how proving an equivalence of singularity categories led to a generalisation of this result.

Matrix Factorisations

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<u>But</u>: Does <u>not</u> agree with experiments!

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But No polynomial r satisfies $r^2 = f$.

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$$\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \right)$$

"(direct) sum" of MFs

is a MFs of f.

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Thm (Auslander–Buchsbaum–Eisenbud–Serre): $\underline{MF}(f) = 0 \iff \{f=0\}$ is a smooth variety

Knörrer's periodicity

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More precisely, let $0 \neq f \in \mathbb{C}[x_0, \dots, x_d]$

Thm (Knörrer 1987) $\quad \underline{\mathrm{MF}}(f) \cong \underline{\mathrm{MF}}(f + y^2 + z^2)$

"up to sums of trivial MFs, there is a bijection between MFs of f and $f + y^2 + z^2$."

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In other words the sequence

$$\underline{MF}(f),\ \underline{MF}(f+y_1^2),\ \underline{MF}(f+y_1^2+y_2^2),\ \underline{MF}(f+y_1^2+y_2^2+y_3^2), \dots$$

is <u>2-periodic</u>. This is called <u>Knörrer's periodicity</u>.

... and beyond.

Singularity categories

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<u>In particular</u>: Singularity categories generalise $\underline{MF}(f)$!

and Knörrer's theorem translates to

$$\mathcal{D}_{sg}\left(\frac{\mathbb{C}[x_0, \dots, x_d, y, z]}{(f + y^2 + z^2)}\right) \cong \mathcal{D}_{sg}\left(\frac{\mathbb{C}[x_0, \dots, x_d]}{(f)}\right)$$

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<u>Q</u>: Can this be extended beyond hypersurfaces

$\mathbb{C}[x_0,\dots,x_d]/(f)$?

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Rem: Proof uses (non-commutative) resolutions of singularities

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$a = 1$: Dong Yang $\Rightarrow$ $$\mathcal{D}_{sg}\left(\frac{\mathbb{C}[x_0, \dots, x_r]}{(x_i x_j - x_k x_\ell \mid i+j = k+\ell)}\right) \cong \mathcal{D}_{sg}\left(\frac{\mathbb{C}[x_1, \dots, x_{r-1}]}{(x_1, \dots, x_{r-1})^2}\right)$$

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$\rightsquigarrow$ recover (r, a) from $K_{r,a}$ $\rightsquigarrow$ recover $R_{r,a}$

Rough Sketch of Proof

$D_{Sg}(R_{r,a})$

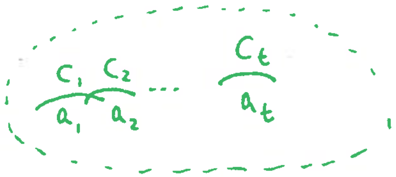
$$\begin{tikzcd} D^b(\mathrm{coh}\,\mathcal{Y}_{r,a}) \arrow[d, two heads] \\ D_{Sg}(R_{r,a}) \end{tikzcd}$$

$a_i \in \mathbb{Z}_{\leq -2}$

min. resolution of $\operatorname{Spec} R_{r,a}$

$$\begin{array}{c} D^b(\operatorname{coh} \mathcal{Y}_{r,a}) \\ \Big\downarrow \\ D_{sg}(R_{r,a}) \end{array}$$

$Y_{r,a}$



$a_i \in \mathbb{Z}_{\leq -2}$

Wemyss's reconstruction algebra

min. resolution of Spec $R_{r,a}$

$$
\begin{tikzcd}
D^b(\mathrm{mod}\, A_{r,a}) \cong D^b(\mathrm{coh}\, Y_{r,a}) \arrow[d, two heads] \\
D_{sg}(R_{r,a})
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$$

tilting VdB, Wemyss

$X_{r,a}$ smooth projective rational surfa

$\mathcal{Y}_{r,a}$ $\quad \underset{a_1}{C_1} \; \underset{a_2}{C_2} \cdots \underset{a_t}{C_t}$ $\quad a_i \in \mathbb{Z}_{\leq -2}$ $\quad$ "compactify" $\rightsquigarrow$ $\quad \underset{a_1}{C_1} \; \underset{a_2}{C_2} \cdots \underset{a_t}{C_t}$

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tilting VdB, Wemyss

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min. resolution of $\operatorname{Spec} R_{r,a}$

exc. collection

$$\begin{tikzcd}
D^b(\mathrm{mod}\, A_{r,a}) \arrow[r, phantom, "\cong"] & D^b(\mathrm{coh}\, Y_{r,a}) \arrow[d] & \mathrm{thick}(\mathcal{O}, \mathcal{O}(-c_1), \dots, \mathcal{O}(-c_1-c_2-\dots-c_t)) \arrow[l] \subset D^b(\mathrm{coh}\, X_{r,a}) \\
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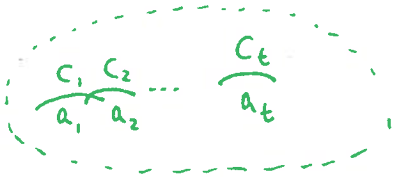
\end{tikzcd}$$

tilting VdB, Wemyss

Hille-Ploog

$\mathcal{Y}_{r,a}$



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 & D_{Sg}(R_{r,a}) & D^b(\mathrm{mod}\, \Lambda_{r,a})
\end{tikzcd}$$

tilting VdB, Wemyss

Hille-Perling

strongly q.h. algebra

$X_{r,a}$ smooth projective rational surface

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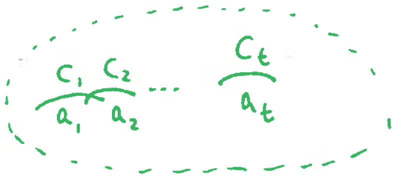

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 & & D^b(\mathrm{mod}\, \Lambda_{r,a}) \arrow[d, two heads] & \\
 & D_{Sg}(R_{r,a}) & D_{Sg}(\quad) &
\end{tikzcd}
$$

$e\Lambda_{r,a}e$

tilting VdB, Wemyss

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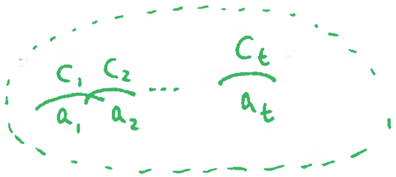

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& & D^b(\mathrm{mod}\, \Lambda_{r,a}) \arrow[d, two heads] & \\
& D_{sg}(R_{r,a}) & D_{sg}(K_{r,a}) &
\end{tikzcd}$$

$K_{r,a} = e\Lambda_{r,a}e$

tilting VdB, Wemyss

Hille-Ploog

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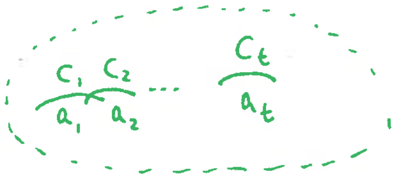

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 & & D^b(\mathrm{mod}\, \Lambda_{r,a}) \arrow[d, twoheadrightarrow] \\
D_{sg}(R_{r,a}) & & D_{sg}(K_{r,a}) \arrow[ll, "\cong"'] \\
 & & e\Lambda_{r,a}e \arrow[u, phantom, "\shortparallel"]
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tilting VdB, Wemyss

induces

Hille-Ploog

strongly q.h. algebra

Focus on strongly q.h.

algebra $\Lambda_{r,a}$

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i.e. $\Lambda_{r,a}$ is a "non-commutative resolution" of $K_{r,a}$

Q: Can we understand Ringel-duality for such endomorphism algebras?

Ringel duality:

translating between western and eastern ways of reading

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[Consider, Rx & Ry]

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Corollary: Let $A = \mathrm{End}_R(\mathrm{SUB}(R))$ be the Auslander algebra of $k[x]/x^n$ (or more generally of a selfinjective Nakayama algebra).

Then

(a) A is left & right strongly q.h.

(b) $\text{Ringel-dual}(A) \cong A \cong A^{op}$