# A surface and a threefold with equivalent singularity categories

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# Introduction

Let $0 \neq f \in \mathbb{C}[\![z_0, \dots, z_d]\!] =: S$

A <u>matrix factorization</u> (MF) of $f$ is a pair

$$(A, B) \in \mathrm{Mat}_n(S) \times \mathrm{Mat}_n(S)$$

satisfying

$$A \cdot B = f \cdot \mathrm{Id}_n = B \cdot A$$

<u>Examples</u>:

$(\mathrm{Id}_n, f\cdot \mathrm{Id}_n)$ and $(f\cdot \mathrm{Id}_n, \mathrm{Id}_n)$

are <u>trivial matrix factorizations</u>

Example: $f = z_0^2 + z_1^2 + z_2^2 + z_3^2$

Step 1: Consider $g = z_0^2 + z_1^2$

$g = (z_0 + iz_1)(z_0 - iz_1) =: XY$ is MF

Step 2:

$$\begin{pmatrix} -X & (z_2 - iz_3) \\ (z_2 + iz_3) & Y \end{pmatrix}, \begin{pmatrix} -Y & (z_2 - iz_3) \\ (z_2 + iz_3) & X \end{pmatrix}$$

is MF for $f$.

## Remark:

(a) A similar approach yields a Matrix fact.

$$A \cdot A = (z_0^2 + z_1^2 + z_2^2 + z_3^2) \cdot Id_4$$

which is used in Dirac's dicovery of the Dirac equation.

(b) More recently, theoretical physicists are interested in understanding and classifying MFs in relation with 2D-Quantum Field Theories.

Example: $f = z_0^2 + z_1^2 + z_2^2 + z_3^2$

Step 1: Consider $g = z_0^2 + z_1^2$

$g = (z_0 + i z_1)(z_0 - i z_1) =: XY$ is MF

Step 2:

$$\begin{pmatrix} -X & (z_2 - i z_3) \\ (z_2 + i z_3) & Y \end{pmatrix}, \begin{pmatrix} -Y & (z_2 - i z_3) \\ (z_2 + i z_3) & X \end{pmatrix}$$

is MF for $f$.

<u>Observation</u> (Knörrer 1987):

Let $0 \neq g \in \mathbb{C}[[z_0, \dots, z_d]]$

The construction in Step 2 defines a map

$$MF(g) \longrightarrow MF\left(g + z_{d+1}^2 + z_{d+2}^2\right).$$

This is a <u>bijection</u> (up to sums of trivial MFs).

More precisely, MFs $(A,B)$ of $f$ can be seen as 2-periodic "curved complexes"

$$\cdots \xrightarrow{A} S^n \xrightarrow{B} S^n \xrightarrow{A} S^n \xrightarrow{B} \cdots$$

In analogy with homotopy categories of complexes one can define the <u>homotopy category of MFs</u> $HMF(f)$.

<u>Thm</u> (<u>Knörrer '87</u>) There is a triangle equiv.

$$HMF(f) \xrightarrow{\sim} HMF(f + z_{d+1}^2 + z_{d+2}^2)$$

# The Buchweitz – Orlov singularity category

$X$ quasi-proj. variety $/\mathbb{C}$

$$\mathrm{Perf}(X) \hookrightarrow D^b(\mathrm{Coh}\, X) \twoheadrightarrow D_{sg}(X) := \frac{D^b(\mathrm{Coh}\, X)}{\mathrm{Perf}(X)}$$

'smooth part' consisting of bounded complexes of vector bundles

Singularity Category measures complexity of singularities of $X$

<u>Thm</u> (Auslander-Buchsbaum & Serre)

$$X \text{ smooth} \iff D_{sg}(X) = 0$$

Thm (Orlov) If $X$ has isolated singularities

$$D_{sg}(X) \cong \bigoplus_{s \in \mathrm{Sing}(X)} D_{sg}(\hat{\mathcal{O}}_s)$$

(up to taking direct summands)

where

$$D_{sg}(R) := \frac{D^b(\text{mod-}R)}{K^b(\text{proj-}R)}$$

<u>singularity category</u> of a noetherian ring $R$.

**Thm** (Buchweitz, Eisenbud)

Let $0 \neq f \in \mathbb{C}[[z_0, \dots, z_d]] =: S$

$$HMF(f) \cong D_{sg}(S/(f))$$

**Def**: Two $\left(\begin{smallmatrix}\text{possibly}\\ \text{noncomm.}\end{smallmatrix}\right)$ algebras $R, S$ are called singular equivalent if there is a triangle equivalence

$$D_{sg}(R) \cong D_{sg}(S)$$

"Trivial examples": If $\operatorname{gldim} R < \infty$ and $\operatorname{gldim} S < \infty$

(a) $D_{sg}(R) \cong 0 \cong D_{sg}(S)$

(b) $D_{sg}(T \times R) \cong D_{sg}(T) \oplus D_{sg}(R) \cong D_{sg}(T \times S)$

(c) $D^b(\text{mod-}A) \cong D^b(\text{mod-}B) \Longrightarrow D_{sg}(A) \cong D_{sg}(B)$

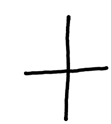
However, for commutative $A$ & $B$ $\Longrightarrow$ $A \cong B$

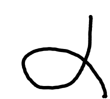
(d) If $A$ and $B$ are commutative and "analytically isomorphic" we can have

$D_{sg}(A) \cong D_{sg}(B)$

e.g. $A = \dfrac{\mathbb{C}[x,y]}{(xy)}$, $B = \dfrac{\mathbb{C}[[x,y]]}{(xy)}$

$\boxed{\text{BUT IN GENERAL}}$ $D_{sg}(A) \not\cong D_{sg}(B)$

e.g. $A = \mathbb{C}[x,y]/(xy)$ 

$B = \dfrac{\mathbb{C}[x,y]}{(y^2-x^2-x^3)}$ 

# Non-trivial singular equivalences between commutative rings

(A) $D_{sg}\left(\dfrac{S}{(f)}\right) \cong D_{sg}\left(\dfrac{S[[y_1,\dots,y_{2n}]]}{(f+y_1^2+\dots+y_{2n}^2)}\right)$

[Knörrer '87]

**Remark** $D_{sg}(R)$ has a (natural) dg-enhancement $D^{dg}_{sg}(R)$ by Drinfeld, Keller.

**Thm** (K. '21, building on Mather-Yau, Knörrer, Orlov, Hua-Keller)

- $R$ complete local Gorenstein with isolated singularity
- $T$ complete local $\mathbb{C}$-algebra, such thath

$$D^{dg}_{sg}(R) \xrightarrow[\text{quasi-equiv.}]{\sim} D^{dg}_{sg}(T)$$

THEN there exists a "Knörrer equiv." $D_{sg}(R) \cong D_{sg}(T)$ (as above)

IF (a) $R \cong S/(f)$ is a hypersurface

OR

(b) $\operatorname{krdim} R \neq \operatorname{krdim} S$

(B) [D. Yang, Y. Kawamata, K.–Karmazyn] all ~ 2015

$$D_{sg}\left( \mathbb{C}[|z_1, z_2|]^{\frac{1}{m}(1,1)} \right) \cong D_{sg}\left( \frac{\mathbb{C}[|z_1, \dots, z_{m-1}|]}{(z_1, \dots, z_{m-1})^2} \right)$$

where $\mathbb{C}[|z_1, \dots, z_d|]^{\frac{1}{m}(a_1, \dots, a_d)}$ is the invariant ring of the following group action:

$$z_i \longmapsto \varepsilon_m^{a_i} z_i \quad \left( \begin{array}{c} \text{for a primitive} \\ m\text{-th root of unity} \\ \varepsilon_m \in \mathbb{C} \end{array} \right)$$

Remarks:

(i) Until recently this was a complete list of non-trivial singular equivalences between comm. rings.

(ii) All these equivalences preserve the parity of the Krull dimension.

(iii) Knörrer's equivalences are the only known examples where both rings have positive Krull dim.

**Thm (K. '21)**

There is a singular equivalence

$$D_{Sg}\left(\mathbb{C}[[z_1, z_2, z_3]]^{\frac{1}{2}(1,1,1)}\right) \cong D_{Sg}\left(\mathbb{C}[[z_1, z_2]]^{\frac{1}{4}(1,1)}\right).$$

Remark:

The Krull dimensions of these invariant rings are 3 and 2, respectively.
In particular, this singular equivalence does <u>not</u> preserve the parity of Krull dimensions.

Proof

__Def__: A noetherian $\mathbb{C}$-algebra $R$ is __syzygy simple__ of __order $m$__ if there exist $S \in \text{mod-}R$ s.th.

(s1) For every $M \in \text{mod-}R$ there is $n \in \mathbb{N}$ s.th. $\Omega^n(M) \in \underline{\text{add}}\text{-}S$.

(s2) $\underline{\text{add}}\text{-}S \cong \text{mod-}\mathbb{C}$

(s3) $\Omega(S) \cong S^{\oplus m}$ in $\underline{\text{mod}}\text{-}R$.

__Prop__: Let $R$ & $S$ be syzygy simple of order $m$. Then

$$D_{sg}(R) \cong D_{sg}(S)$$

<u>Examples:</u> (a) $K_m := \frac{\mathbb{C}[z_1, \dots, z_m]}{(z_1, \dots, z_m)^2}$

is syzygy simple of order $m$,

with $S = \frac{\mathbb{C}[z_1, \dots, z_m]}{(z_1, \dots, z_m)}$ simple

(b) [cf. Iyama-Wemyss, "GL(2) McKay-Corr."]

$$R_m := \mathbb{C}[[z_1, z_2]]^{\frac{1}{(m+1)}(1,1)}$$

is syzygy simple of order $m$.

(c) $R := \mathbb{C}[[z_1, z_2, z_3]]^{\frac{1}{2}(1,1,1)}$

is syzygy simple of order 3.

Here, $S = \Omega(\omega_R)$.

[uses Auslander-Reiten theory]

In combination with the proposition above this shows all non-trivial singular equivalences between comm. rings except for Knörrer's

# Proof of Proposition

# Blackbox (Heller):

$\mathcal{A}$ category with $E : \mathcal{A} \longrightarrow \mathcal{A}$

there is a universal way of "enlarging" $\mathcal{A}$ s.th. $E$ "becomes" autoequivalence

$$S(\mathcal{A}, E)$$

"stabilization" of $(\mathcal{A}, E)$

**Thm** [Keller-Vossieck '87]
R noetherian. There is a $\Delta$-equiv.

$$S(\underline{\mathrm{mod}}\text{-}R, \Omega) \cong D_{sg}(R)$$

it is the "stabilization" of

$$F: \underline{\mathrm{mod}}\text{-}R \longrightarrow D_{sg}(R).$$

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**Lemma**: If R syzygy simple

$$(\underline{\mathrm{add}}\text{-}S, \Omega) \subset (\underline{\mathrm{mod}}\text{-}R, \Omega)$$

is a left triangulated subcategory and the "stabilization" is a $\Delta$-equiv.

$$S(\underline{\mathrm{add}}\text{-}S, \Omega) \cong S(\underline{\mathrm{mod}}\text{-}R, \Omega)$$

Since $\underline{\mathrm{add}}\text{-}S \cong \mathrm{mod}\text{-}\mathbb{C}$ is semisimple there is a unique left triangulated structure on

$$(\underline{\mathrm{add}}\text{-}S, \Omega)$$

which is completely determined by

$$\Omega(S) \cong S^{\oplus m}.$$

Summing up, the Proposition follows from

$$\begin{tikzcd}
 & \mathcal{S}(\underline{\mathrm{add}}\text{-}S_R, \Omega_R) \arrow[r, phantom, "\cong"] \arrow[dl, phantom, "\cong"] & \mathcal{S}(\underline{\mathrm{add}}\text{-}S_S, \Omega_S) \arrow[d, phantom, "\cong"] & \\
\mathcal{S}(\underline{\mathrm{mod}}\text{-}R, \Omega_R) \arrow[d, phantom, "\cong"] & & \mathcal{S}(\underline{\mathrm{mod}}\text{-}S, \Omega_S) \arrow[d, phantom, "\cong"] \\
D_{sg}(R) & & D_{sg}(S)
\end{tikzcd}$$

More details on examples of syzygy simple algebras

<u>Lemma</u>: If $R$ is a local Cohen-Macaulay ring of Krull dimension $d$ and $M \in \text{mod-}R$ then for all $n \geq d$

$$\Omega^n(M) \text{ is a}$$

maximal Cohen-Macaulay $R$-module

<u>Thm</u>: The invariant rings

$Q = \mathbb{C}[[z_1, \dots, z_d]]^G$, $G \subset GL(d, \mathbb{C})$ finite

are Cohen-Macaulay

<u>Cor</u>: Understanding high syzygies

$$\Omega^{\gg 0}(M), \quad M \in \text{mod-}R$$

reduces to understanding syzygies of maximal Cohen-Macaulay modules $MCM(R)$. (which are again MCM!)

<u>Lemma</u>: $Q$ invariant ring as above.
Then there is an injective object
$\omega_Q \in MCM(Q)$ "canonical module"
In particular, for all $M \in MCM(Q)$:

$$0 \to \mathrm{Hom}_Q(M, \omega_Q) \to \mathrm{Hom}_Q(Q^n, \omega_Q) \to \mathrm{Hom}_Q(\Omega(M), \omega_Q) \to 0$$

is exact.

$\rightsquigarrow$ All morphisms
factor

$$\begin{tikzcd} \Omega(M) \arrow[rr] \arrow[dr] & \circlearrowleft & \omega_Q \\ & Q^n \arrow[ur] & \end{tikzcd}$$

# Irreducible morphisms in $MCM(R)$:

[Auslander-Reiten]

$$\begin{tikzcd} & R \arrow[dr, shift left=2] \arrow[dr] \arrow[dr, shift right=2] & \\ M \arrow[ur, shift left=2] \arrow[ur] \arrow[ur, shift right=2] & & \omega_R \arrow[ll, shift left=2] \arrow[ll] \arrow[ll, shift right=2] \end{tikzcd}$$

$$R = \mathbb{C}[z_1, z_2, z_3]^{\frac{1}{2}(1,1,1)}$$

$\Rightarrow \Omega(N) \in \underline{\mathrm{add}}_R(M)$ for all $N \in MCM(R)$

and $\underline{\mathrm{End}}_R(M) \cong \mathbb{C}$

$\Rightarrow$ $\boxed{R \text{ is syzygy simple}}$

Applying $\Omega(-)$ to

$$0 \longrightarrow R \longrightarrow \omega_R^{\oplus 3} \longrightarrow M \longrightarrow 0$$

shows $\boxed{\Omega(M) \cong \Omega(\omega_R^{\oplus 3}) \cong M^{\oplus 3}}$

$$R_m := \mathbb{C}[[z_1, z_2]]^{\frac{1}{(m+1)}(1,1)}$$

$$\begin{tikzcd}
 & N_1 \arrow[r, shift left] \arrow[r, shift right] & N_2 \arrow[d, shift left] \arrow[d, shift right] \\
S & & \cdots \arrow[d, shift left] \arrow[d, shift right] \\
\omega_{R_m} \arrow[u, shift left] \arrow[u, shift right] & & N_{m-2} \arrow[ll, shift left] \arrow[ll, shift right]
\end{tikzcd}$$

$$\boxed{\text{stable Auslander-Reiten quiver}}$$

$\rightsquigarrow$ $\exists$ morphisms $N_i \xrightarrow{\neq 0} \omega_{R_m}$ in $\underline{MCM}(R_m)$

& $\omega_{R_m} \xrightarrow{id \neq 0} \omega_{R_m}$

lemma $\rightsquigarrow$ $\Omega(M) \in \underline{add}\text{-}S$ for $M \in MCM(R_m)$

Moreover, $\underline{End}_{R_m}(S) \cong \mathbb{C}$

$\longrightarrow$ $\boxed{R_m \text{ syzygy simple}}$ (of order $m$ )
[uses K-theory]