# Matrix factorisations: Knörrer's periodicity and beyond.

Martin Kalck
University of Edinburgh

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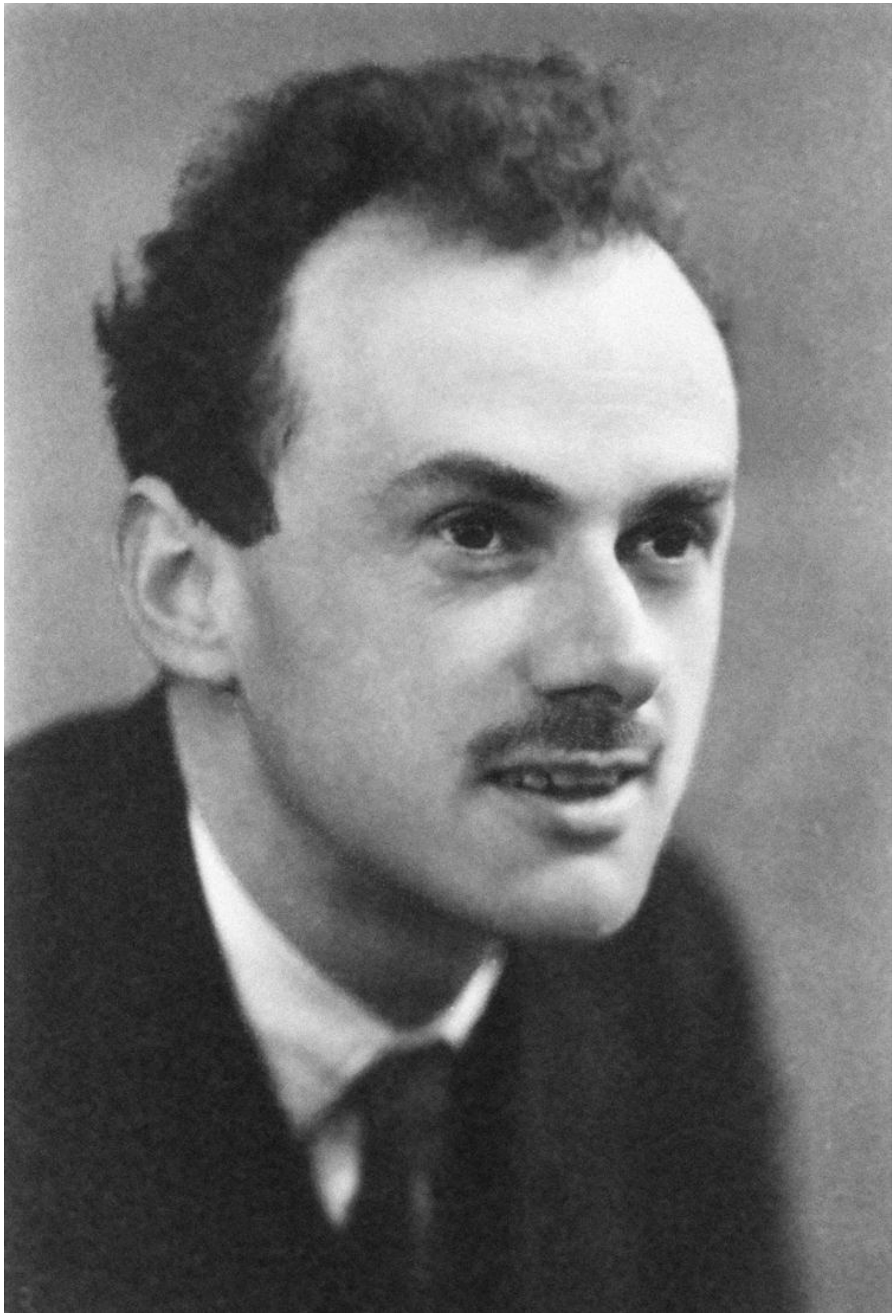
<u>But</u>: Does <u>not</u> agree with experiments!



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Hope (Dirac) Find square matrix $A$ such that $A^2 = f \cdot \mathrm{Id}$.

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Indeed

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Id}$$

$$i \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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with $D^2 = (f + y^2 + z^2)\,\mathrm{Id} = (x_1^2 + x_2^2 + y^2 + z^2)\,\mathrm{Id}$

Using this Dirac can factor Klein-Gordon as

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There is a physical interpretation of Dirac's 'mathematical trick' leading to prediction of spin, antimatter... and a Nobel prize for Dirac in 1933.

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Q: When do non-trivial MFs exist?

Answer involves geometry!

# More precisely

<u>Thm</u> (Eisenbud; building on Auslander – Buchsbaum – Serre)

All MFs of $f$ are <u>sums of trivial MFs</u> iff

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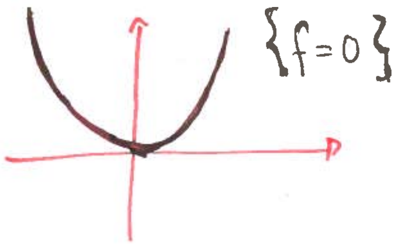


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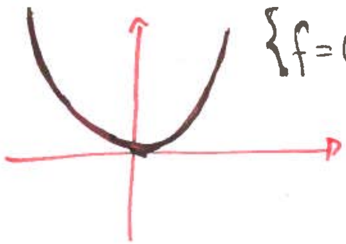

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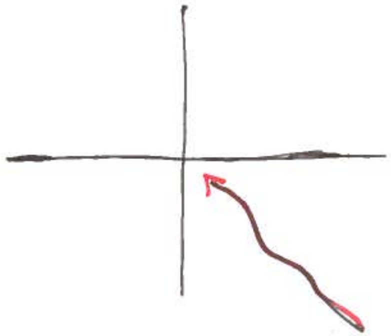

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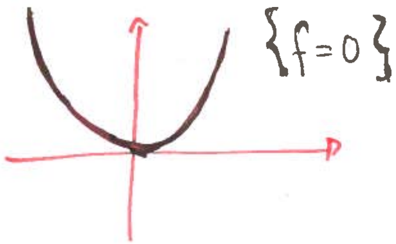


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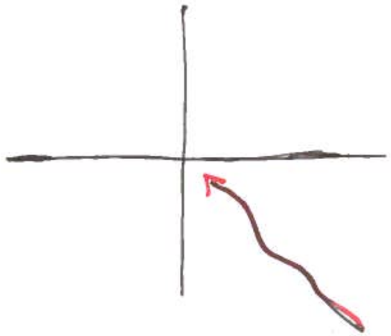

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<u>And</u> $(x+iy,\, x-iy)$, $(x-iy,\, x+iy)$ are <u>non-trivial</u> MFs

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<u>Def</u>: <u>MF</u>$(f)$ := collection of all MFs of $f$

<u>modulo</u> sums of trivial MFs $(1,f), (f,1)$.

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Thm above shows: $\underline{MF}(f) = 0 \iff \{f=0\}$ smooth.

# Knörrer's periodicity

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**Thm (Knörrer 1987)** $\underline{MF}(f) \cong \underline{MF}(f + y^2 + z^2)$

"up to sums of trivial MFs, there is a bijection between MFs of $f$ and $f + y^2 + z^2$."

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In other words the sequence

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is <u>2-periodic</u>. This is called <u>Knörrer's periodicity</u>.

... and beyond.

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In particular: Singularity cat. generalise $\mathrm{MF}(f)$ !

and Knörrer's theorem translates to

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Q: Can this be extended beyond hypersurfaces $S/f$ ?

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Obs. (Yang) $\mathcal{D}_{Sg}(R_n) \cong \mathcal{D}_{Sg}(K_n)$

this is a finite dimensional algebra!

__Rem__: $R_n$ is coord. ring of a cyclic quotient singularity $\mathbb{C}^2/G$, where $G \cong \mathbb{Z}/n\mathbb{Z}$.

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<u>Knörrer</u>: $\mathcal{D}_{sg}(A_n) \cong \mathcal{D}_{sg}\left(A_n/(y,z) \cong \frac{k[x]}{x^n}\right)$

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finite dimensional (non-commutative) algebra

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- $K_{n,a}$ admits explicit presentation using generators and relations — closely related to commutative $R_{n,a}/(x^n, y^n)$

# Equivalences of singularity categories

# via (non-commutative) resolutions

Key idea

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We combine these ideas with a construction of Hille & Ploog to obtain the desired equivalence of singularity categories.

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min.
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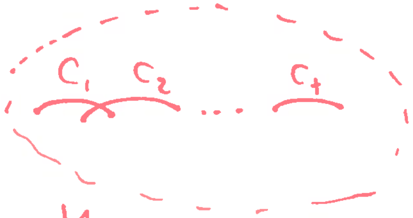
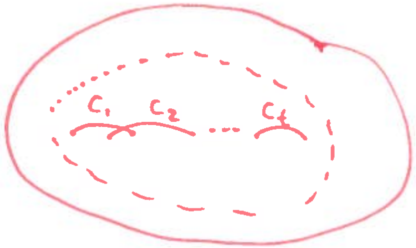



$Y$

$$\xhookrightarrow[\text{flat \& affine open immersion}]{\nu}$$




$X$ smooth proj. rational surface

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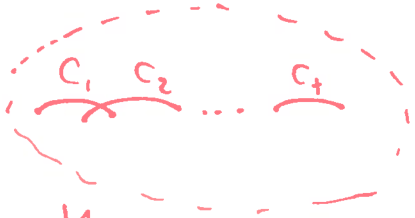
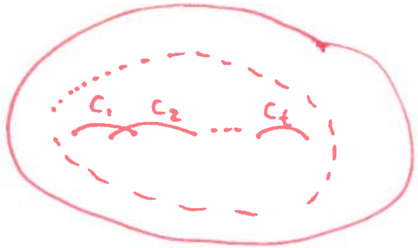



$Y \xhookrightarrow{\nu}$ flat & affine open immersion

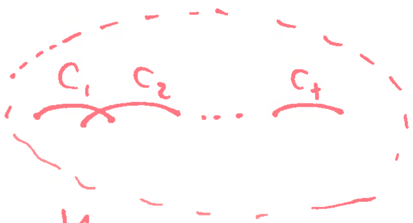



$X$ smooth proj. rational surface

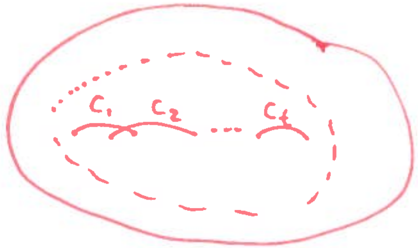
(i.e. $H^i(X, \mathcal{O}_X) = 0 \ \forall i > 0$)

$$\mathcal{X} = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$$

$\uparrow \pi$ min. res. of sing.




$\mathcal{Y}$ $\xhookrightarrow{\nu}$ (flat & affine open immersion)

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$$D^b(\operatorname{coh} \mathcal{Y}) \xleftarrow{\nu^*} D^b(\operatorname{coh} X)$$

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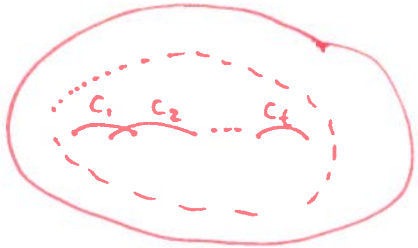
$Y \xhookrightarrow{\nu} X$

flat & affine open immersion

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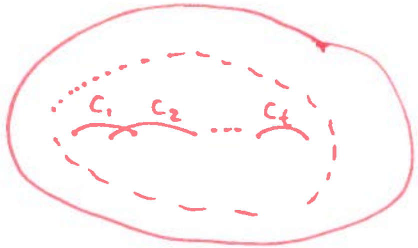


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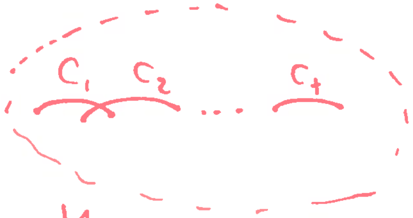
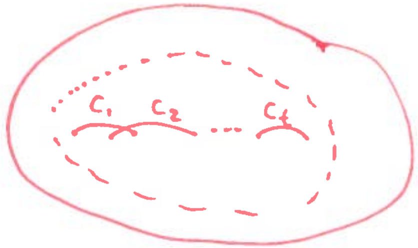



$Y \xhookrightarrow{\iota} \mathcal{X}$

flat & affine open immersion




$\mathcal{X}$ smooth proj. rational surface (i.e. $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0 \ \forall i > 0$)

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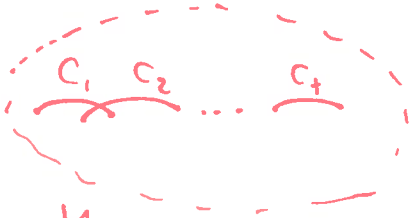
exc. seq.

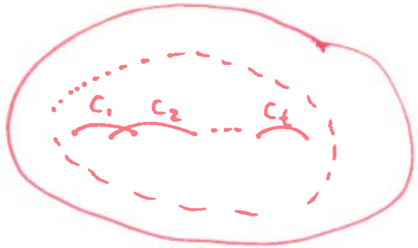
$\cong$ Hille–Ploog

$D^b(A\text{-mod})$

$D^b(\mathrm{coh}\, Y) \xrightarrow{\pi_*} \mathcal{D}_{sg}(X)$

$$X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$$




$$\begin{tikzcd}
X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}} & & \\
Y \arrow[u, "\pi"', "\text{min. res. of sing.}"] \arrow[r, hook, "\nu"', "\text{flat \& affine open immersion}"] & \mathcal{X} & \\
D^b(\operatorname{coh} Y) \arrow[d, "\pi_*"] & D^b(\operatorname{coh} \mathcal{X}) \arrow[l, "\nu^*"'] & \\
D_{sg}(\cancel{X}) & &
\end{tikzcd}$$

$\mathcal{X}$ smooth proj. <u>rational</u> surface (i.e. $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0 \ \forall i > 0$)

$$D^b(\operatorname{coh} \mathcal{X}) \supset \operatorname{thick}\big(\underbrace{\mathcal{O}, \mathcal{O}(-C_1), \dots, \mathcal{O}(-C_1 - C_2 - \dots - C_t)}_{\text{exc. seq.}}\big)$$

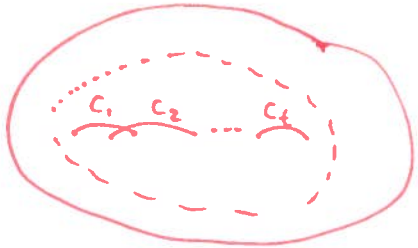
$\cong$ Hille–Ploog

$D^b(A\text{-mod})$ ← fin. dim. noncomm. algebra

$X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$

$\uparrow \pi$ min. res. of sing.


$\xhookrightarrow{\nu}$ flat & affine open immersion




$X$ smooth proj. rational surface (i.e. $H^i(X, \mathcal{O}_X) = 0 \ \forall i > 0$)

$$D^b(\operatorname{coh} Y) \xleftarrow{\nu^*} D^b(\operatorname{coh} X) \supset \operatorname{thick}(\mathcal{O}, \mathcal{O}(-C_1), \dots, \mathcal{O}(-C_1 - C_2 - \dots - C_t))$$

exc. seq.

$\cong$ Hille–Ploog

$D^b(A\text{-mod})$ ← fin. dim. noncomm. algebra

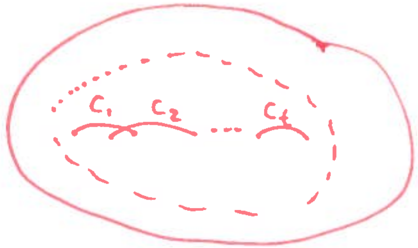
$$\begin{tikzcd}
D^b(\operatorname{coh} Y) \arrow[d, "\pi_Y"] & D^b(A\text{-mod}) \arrow[d] \\
D_{sg}(X) & D_{sg}(eAe)
\end{tikzcd}$$

$$X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$$

$$\uparrow \pi$$

min. res. of sing.


$$\xhookrightarrow[\text{flat \& affine open immersion}]{\nu}$$




$X$ smooth proj. rational surface (i.e. $H^i(X,\mathcal{O}_X)=0\ \forall i>0$)

$$\begin{tikzcd}
D^b(\operatorname{coh} Y) \arrow[d, "\pi_*"'] \arrow[dr] & & D^b(\operatorname{coh} X) \arrow[ll, "\nu^*"'] \supset \operatorname{thick}(\mathcal{O}, \mathcal{O}(-C_1), \dots, \mathcal{O}(-C_1-C_2-\dots-C_t)) \\
\arrow[r, phantom, "\circlearrowleft"] & \dfrac{D^b(Y)}{\operatorname{thick}(\mathcal{O}_Y)} \arrow[dl] & \\
D_{sg}(X) & &
\end{tikzcd}$$

exc. seq.

$$\operatorname{thick}(\mathcal{O}, \dots, \mathcal{O}(-C_1-\dots-C_t)) \cong D^b(A\text{-mod}) \quad \text{Hille-Ploog}$$

$A$ ← fin. dim. noncomm. algebra

$$D^b(A\text{-mod}) \to D_{sg}(eAe)$$

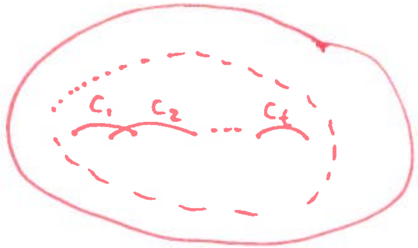
$$X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$$

$$\begin{tikzcd} X \\ Y \arrow[u, "\pi"] \end{tikzcd}$$

min. res. of sing.


$$\begin{tikzcd} Y \arrow[r, hook, "\nu"] & X \end{tikzcd}$$

flat & affine open immersion




$X$ smooth proj. rational surface (i.e. $H^i(X, \mathcal{O}_X) = 0 \ \forall i > 0$)

$$\begin{tikzcd}
D^b(\operatorname{coh} Y) \arrow[d, "\pi_*"'] \arrow[dr, two heads] & & \mathcal{D}^b(\operatorname{coh} X) \arrow[ll, "\nu^*"'] & \supset \operatorname{thick}\left(\mathcal{O}, \mathcal{O}(-C_1), \dots, \mathcal{O}(-C_1 - C_2 - \dots - C_t)\right) \arrow[d, phantom, "\cong"] \\
{} \arrow[r, phantom, "\circlearrowleft"] & \dfrac{D^b(Y)}{\operatorname{thick}(\mathcal{O}_Y)} \arrow[dl, two heads] & \dfrac{D^b(A)}{\operatorname{thick}(Ae)} \arrow[dr, two heads] & D^b(A\text{-mod}) \arrow[l, two heads] \arrow[d, two heads] \\
\mathcal{D}_{sg}(X) & & & \mathcal{D}_{sg}(eAe)
\end{tikzcd}$$

exc. seq.

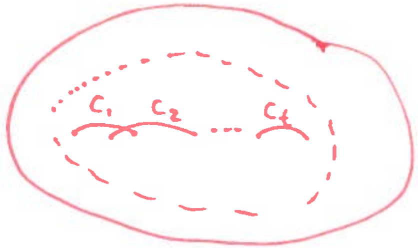
Hille – Ploog

fin. dim. noncomm. algebra

$$X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$$

$\uparrow \pi$ min. res. of sing.


$$Y \xrightarrow[\text{flat \& affine open immersion}]{\nu} X$$




$X$ smooth proj. rational surface (i.e. $H^i(X, \mathcal{O}_X) = 0 \ \forall i > 0$)

$$D^b(\operatorname{coh} Y) \xleftarrow{\nu^*} D^b(\operatorname{coh} X) \supset \operatorname{thick}\big(\mathcal{O}, \mathcal{O}(-C_1), \dots, \mathcal{O}(-C_1 - C_2 - \dots - C_t)\big)$$

exc. seq.

$\cong$ Hille–Ploog

$$D^b(A\text{-mod}) \leftarrow \text{fin. dim. noncomm. algebra}$$

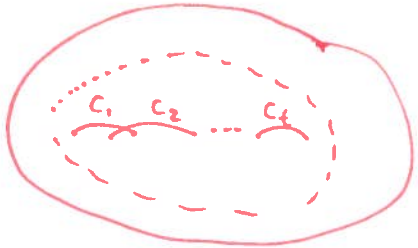
$$\begin{tikzcd}
D^b(\operatorname{coh} Y) \arrow[d, "\pi_*"'] \arrow[dr] & & & D^b(A\text{-mod}) \arrow[dl] \arrow[d] \\
\arrow[r, phantom, "\circlearrowright"] & D^b(Y)/\operatorname{thick}(\mathcal{O}_Y) \arrow[dl] & D^b(A)/\operatorname{thick}(Ae) \arrow[l, dashed, "\cong"] \arrow[dr] & \\
D_{sg}(X) & & & D_{sg}(eAe)
\end{tikzcd}$$

$\circlearrowright$

$$X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$$

$$\uparrow \pi \quad \text{min. res. of sing.}$$


$$\xhookrightarrow[\text{flat \& affine open immersion}]{\nu}$$




$X$ smooth proj. <u>rational</u> surface (i.e. $H^i(X, \mathcal{O}_X) = 0 \ \forall i > 0$)

$$D^b(\operatorname{coh} Y) \xleftarrow{\nu^*} D^b(\operatorname{coh} X) \supset \operatorname{thick}\big(\mathcal{O}, \mathcal{O}(-C_1), \dots, \mathcal{O}(-C_1 - C_2 - \dots - C_t)\big)$$

exc. seq.

$\cong$ Hille–Ploog

$D^b(A\text{-mod})$ ← fin. dim. noncomm. algebra

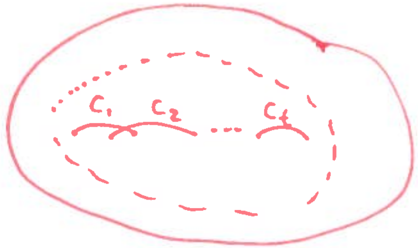
$$\begin{tikzcd}
D^b(\operatorname{coh} Y) \arrow[dd, "\pi_*"'] \arrow[dr, two heads] \arrow[ddr, phantom, "\circlearrowleft"] & & & D^b(A\text{-mod}) \arrow[dl, two heads] \arrow[dd] \\
& \dfrac{D^b(Y)}{\operatorname{thick}(\mathcal{O}_Y)} \arrow[dl] \arrow[r, phantom, "\circlearrowleft"] & \dfrac{D^b(A)}{\operatorname{thick}(Ae)} \arrow[l, dashed, "\cong"'] \arrow[dr] & \\
D_{sg}(X) & & & D_{sg}(eAe) \arrow[lll, dashed, "\exists \ \cong"]
\end{tikzcd}$$

$\Downarrow$ formal arguments

$X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$

$\uparrow \pi$ min. res. of sing.


$\xhookrightarrow[\text{flat \& affine open immersion}]{\nu}$




$X$ smooth proj. rational surface (i.e. $H^i(X, \mathcal{O}_X) = 0 \ \forall i > 0$)

$$
\begin{tikzcd}
D^b(\operatorname{coh} Y) \arrow[d, "\pi_*"'] \arrow[dr] & & D^b(\operatorname{coh} X) \arrow[ll, "\nu^*"'] & \supset \operatorname{thick}(\mathcal{O}, \mathcal{O}(-C_1), \dots, \mathcal{O}(-C_1 - C_2 - \dots - C_t)) \arrow[d, "\cong", "\text{Hille–Ploog}"'] \\
& D^b(Y)/\operatorname{thick}(\mathcal{O}_Y) \arrow[dl] & D^b(A)/\operatorname{thick}(Ae) \arrow[l, dashed, "\cong"'] \arrow[dr] & D^b(A\text{-mod}) \arrow[l] \arrow[d] \\
D_{sg}(X) & & & D_{sg}(eAe) \arrow[lll, dashed, "\exists \cong"]
\end{tikzcd}
$$

exc. seq.

fin. dim. noncomm. algebra

$\Downarrow$ formal arguments

$\mathcal{K} = D_{sg}(eAe)$ Knörrer inv. alg.

$$X = \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$$

$\uparrow \pi$ min. res. of sing.

$$Y \supset C_1 \cup C_2 \cup \dots \cup C_t \xhookrightarrow[\text{flat \& affine open immersion}]{v} \mathcal{X} \supset C_1 \cup C_2 \cup \dots \cup C_t$$

$\mathcal{X}$ smooth proj. rational surface (i.e. $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0 \ \forall i > 0$)

$$\begin{tikzcd}
D^b(\operatorname{coh} Y) \arrow[d, "\pi_*"'] \arrow[dr] & & D^b(\operatorname{coh} \mathcal{X}) \arrow[ll, "v^*"'] \supset \operatorname{thick}\big(\mathcal{O}, \mathcal{O}(-C_1), \dots, \mathcal{O}(-C_1 - C_2 - \dots - C_t)\big) \arrow[d, phantom, "\cong"] \\
 \arrow[r, phantom, "\circlearrowright"] & D^b(Y)/\operatorname{thick}(\mathcal{O}_Y) \arrow[dl] & D^b(A\text{-mod}) \arrow[dl] \arrow[d] \\
D_{sg}(X) & D^b(A)/\operatorname{thick}(Ae) \arrow[ul, dashed, "\cong"'] \arrow[dr] & \\
 & & D_{sg}(eAe) \arrow[lll, dashed, "\exists\ \cong"']
\end{tikzcd}$$

exc. seq.

Hille – Ploog

$A$: fin. dim. noncomm. algebra

$\cong$ formal arguments

Knörrer inv. alg.

$$eAe \cong \operatorname{End}_{\mathcal{X}}(V)$$

$V$: vector bundle