# Spherical subcategories and new invariants for triangulated categories

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This is joint work with A. Hochenegger & D. Ploog:

*Spherical subcategories in algebraic geometry*, arXiv:1208.4046.
*Spherical subcategories in representation theory*, in preparation.

# Setup and Motivation

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$\Longrightarrow$ **Twist-functor** $T_X \colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ *is* **equivalence** *(Seidel & Thomas)*

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*Dimension* $= 2$*:* **spherelike objects** – *would like to develop general theory.*

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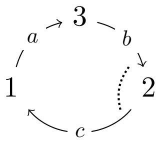

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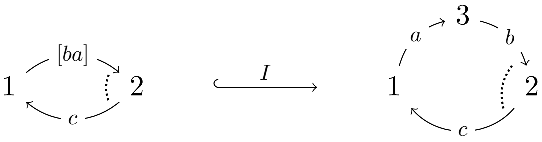

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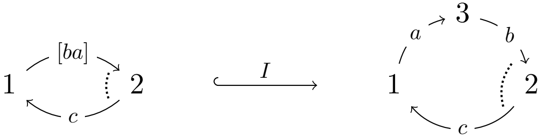

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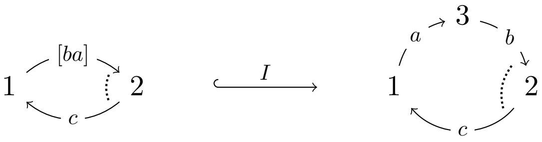

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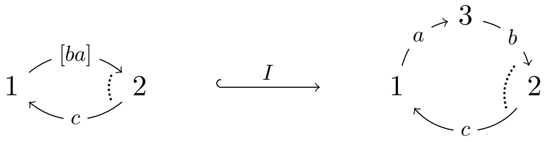

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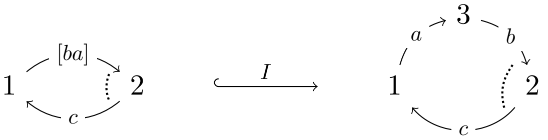

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*In general,* $\mathcal{T}_X$ *much bigger than* $\langle X\rangle_{\mathcal{T}}$ *– triangulated subcategory generated by* $X$*.*

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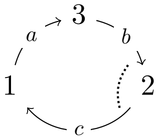

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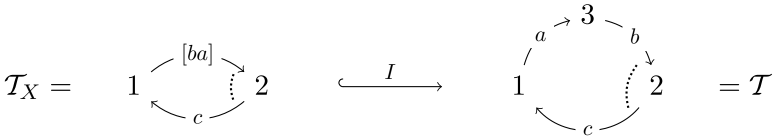

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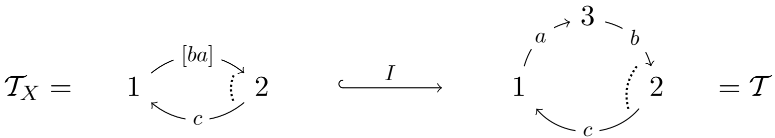

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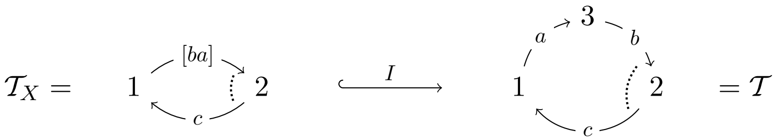

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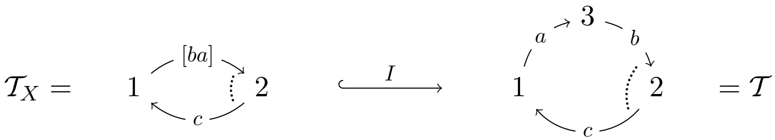

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Can show $\mathcal{D}^b(Y)_{\mathcal{O}_Y} \cong \mathcal{D}^b(X)$

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- $\mathcal{S}p(\mathcal{T}) := \{\mathcal{T}_X \mid X$ spherelike (and indecomposable) $\}$
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## Remark

*Derive* **numerical invariants***, e.g. height, cardinality & width of these posets.*

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## Example

$C$ smooth projective **curve** over $k$, then $\mathcal{S}p_1(\mathcal{D}^b(C)) = \{\mathcal{D}^b(C)\}$.

# Hereditary examples

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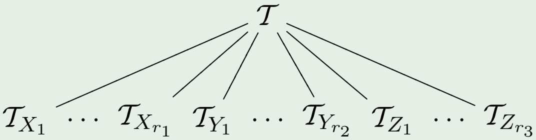

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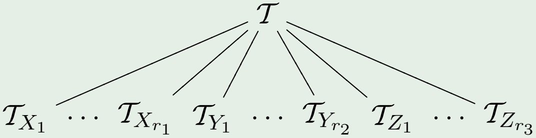

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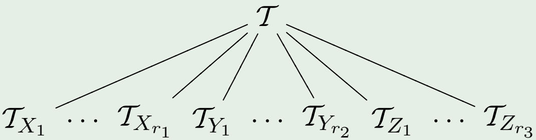

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## Remark

*if $Q$ has full Euclidean subquiver $\Rightarrow \mathcal{S}p(\mathcal{T}) \neq \emptyset$.*

# Derived discrete examples

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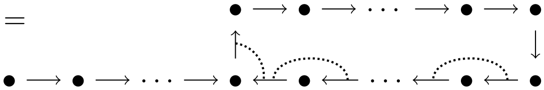

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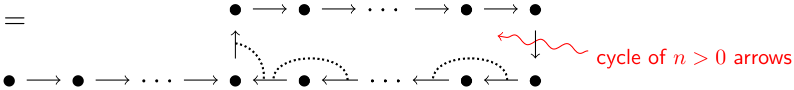

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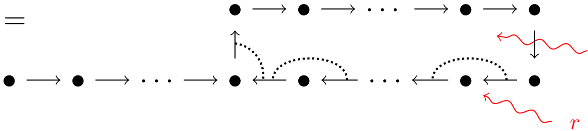

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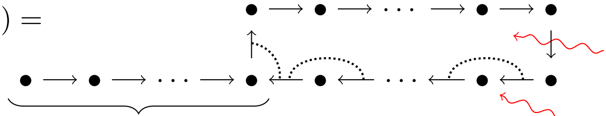

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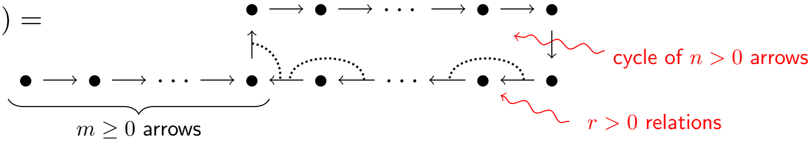

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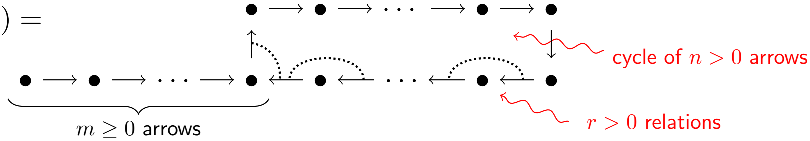

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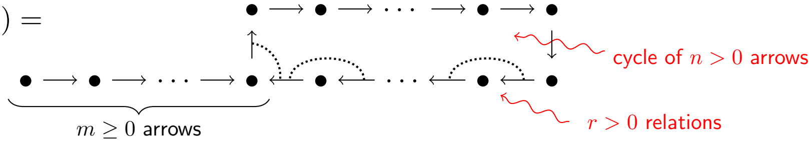

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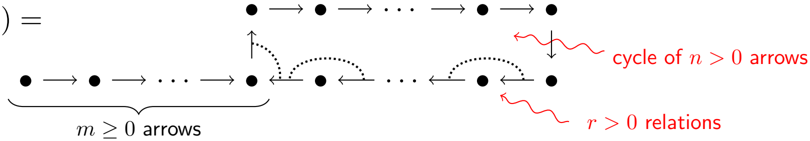

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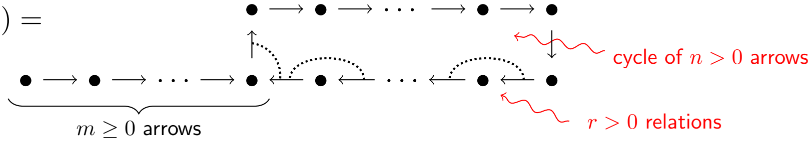

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**Remark**

$$|\,\mathcal{S}p_d(\mathcal{T})\,| = \begin{cases} m+r & \textit{if } d=1-r \\ n-r & \textit{if } d=1+r \\ 0 & \textit{otherwise.} \end{cases}$$

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$\Rightarrow$ *the* **parameters** $r$, $n$ *and* $m$ *are* **determined by** $\{|\,\mathcal{S}p_d(\mathcal{T})\,|\}_{d \in \mathbb{Z}}$.

*In other words, this sequence is a* **complete derived invariant** *for* **Vossieck's discrete derived algebras** $\Lambda(r, n, m)$.

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- $|\, \overline{\mathcal{S}p}(\mathcal{T}) \,| > 2$ $\Rightarrow$ $\mathcal{T}$ *is not* **Vossieck-discrete**.
- height $\mathcal{S}p(\mathcal{T}) > 2$ $\Rightarrow$ $\mathcal{T}$ *is not* **Vossieck-discrete**.
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# Posets of arbitrary height

Auslander algebra of $k[x]/(x^n)$:

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$+$ Relations

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## Fact

*The corresponding poset $\mathcal{S}p(\mathcal{T})$ has* **height** $\geq n-1$.

# Posets of infinite cardinality and width

$$\mathcal{T} = \mathcal{D}^b(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathcal{D}^b\Big( \begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] & 2 \end{tikzcd} \otimes_k \begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] & 2 \end{tikzcd} \Big)$$

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**Fact**

$\mathcal{S}p(\mathcal{T})$ *has infinite cardinality and infinite width.*

# Summary and questions

$\boxed{X \in \mathcal{T}\ d\text{-}\textbf{spherelike}} \rightsquigarrow$

$\mathcal{T}_X \subset \mathcal{T}$ **spherical subcategory**
i.e. **unique maximal** $\Delta -$ subcategory,
s.th.$X \in \mathcal{T}_X$ is $d -$ **spherical.**

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## Questions

- *Are there* **relations to other invariants** *of triangulated categories?*

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## Questions

- *Are there* **relations to other invariants** *of triangulated categories?*
- *When are height, cardinality or width finite? What are* **good bounds***?*

Thank you !