Relative Singularity Categories

Martin Kalck

University of Bonn, Germany

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Introduction

Idea (Van den Bergh)

Replace $\mathcal{D}^b(Y)$ *by* $\mathcal{D}^b(A)$ *for a "nice" algebra* A *(e.g.* $\mathrm{gl.dim}(A) < \infty$*) and consider it as* **categorical resolution** *of* X *if there is an embedding*

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  where $n = n(S_i)$ is given by the length of the τ–orbit of M_i, by [KY].

A natural question

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The implication $(ii) \Rightarrow (i)$ holds more generally for arbitrary NCRs A and A' of arbitrary isolated Gorenstein singularities R and R'.

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Remark

Knörrer's Periodicity yields a wealth of non-trivial examples for (i):

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+xy)\big),$$

where $S = k[\![z_0, \ldots, z_d]\!]$, $f \in (z_0, \ldots, z_d)$ and $d \geq 0$.

Example

Let $R = \mathbb{C}[\![x]\!]/(x^2)$ and $R' = \mathbb{C}[\![x,y,z]\!]/(x^2+yz)$. **Knörrer's equivalence** and our theorem above yield a triangle equivalence

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The quiver algebra on the right hand side is the **completion** of the preprojective algebra of the Kronecker quiver $\Pi\big(\circ \rightrightarrows \circ\big)$.

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Key Statement (Kalck & Yang)

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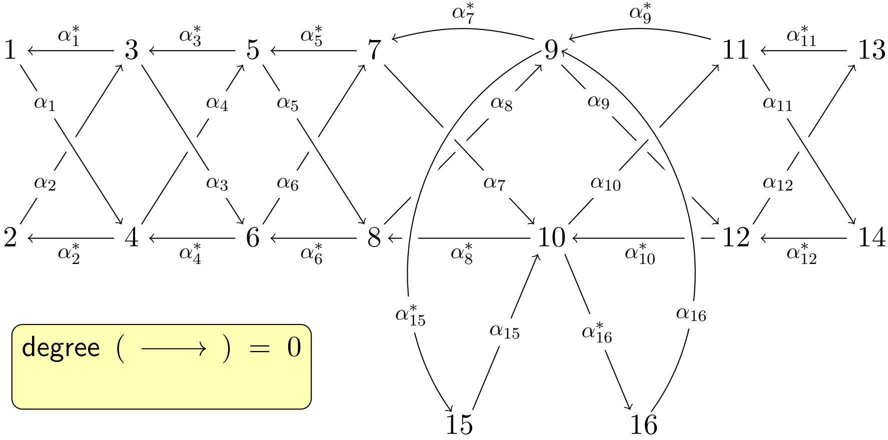
Corollary

$\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$ *implies* $\Delta_R(\mathsf{Aus}(R)) \cong \Delta_{R'}(\mathsf{Aus}(R'))$.

Example: The dg-Auslander algebra of an odd-dimensional $\mathbb{E}_8$-singularity: $f = z_0^3 + z_1^5 + z_2^2 + \ldots + z_d^2$



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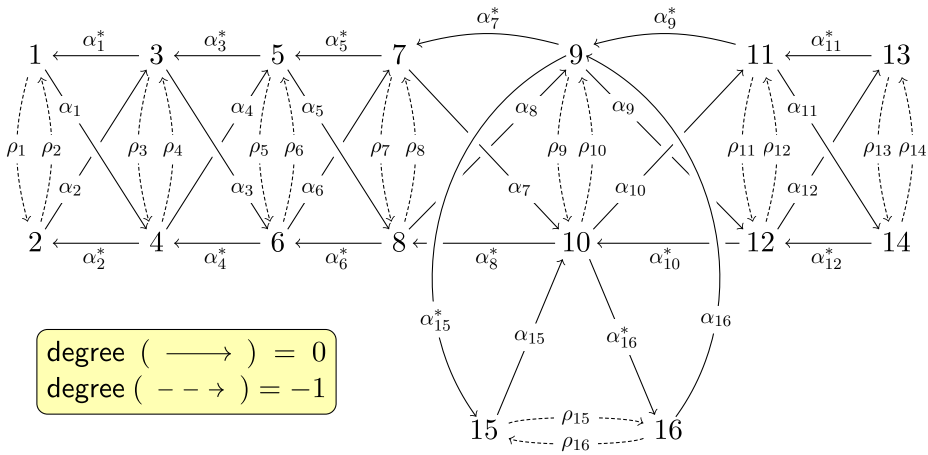

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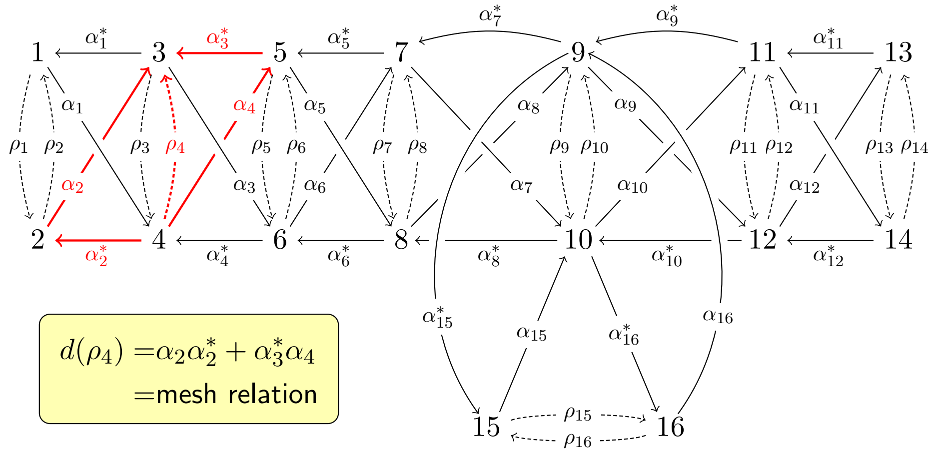

$$d(\rho_4) = \alpha_2\alpha_2^* + \alpha_3^*\alpha_4$$
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Let A be a Noetherian ring and let $e \in A$ be an idempotent. The exact functor $\mathrm{Hom}_A(eA, -)$ induces an equivalence of triangulated categories

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A "purely commutative" application

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- Wunram's Theorem shows that SCM(R) is **representation-finite.** Let $M_0 = R, M_1, \dots, M_t$ be the set of **indecomposable** special MCMs. The **reconstruction algebra** $\Lambda := \mathrm{End}_R\big(\bigoplus_i M_i\big)$ has gl. dim ≤ 3, by work of Wemyss.
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In general, R is **not Gorenstein**. In that case, $\mathrm{MCM}(R)$ is **not Frobenius**. Moreover, the singularity category is **not Krull-Schmidt**.

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We may take the **stable category** $\underline{\mathrm{SCM}}(R)$. If R is Gorenstein, then $\underline{\mathrm{SCM}}(R) \cong \underline{\mathrm{MCM}}(R) \cong \mathcal{D}_{sg}(R)$.

Description of the stable category $\underline{\underline{\mathrm{SCM}}}(R)$

This is joint work with Osamu Iyama, Michael Wemyss & Dong Yang.

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- $\mathsf{gl.\,dim}(\Lambda) < \infty \qquad \Longrightarrow \qquad \mathcal{D}_{sg}(e\Lambda e) \cong \underline{\underline{\mathsf{SCM}}}(R)$

Example

Let $G \subseteq \mathrm{GL}(2,\mathbb{C})$ be the cyclic group of order 27 generated by

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{19} \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}),$$

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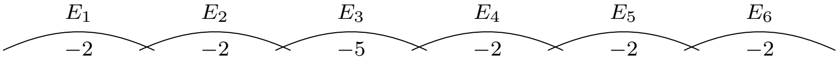

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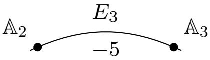

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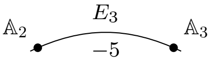

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Our Theorem yields a description of the stable category of SCMs:

$$\underline{\underline{\mathrm{SCM}}}(R_{27,19}) \cong \mathcal{D}_{sg}(X) \cong \underline{\mathrm{MCM}}\left(\frac{\mathbb{C}[\![x,y,z]\!]}{(x^3+yz)}\right) \oplus \underline{\mathrm{MCM}}\left(\frac{\mathbb{C}[\![x,y,z]\!]}{(x^4+yz)}\right)$$

Thank you!