Singularity categories of categorical resolutions of Kodaira cycles and gentle algebras

Martin Kalck

University of Bonn, Germany

ICRA XIV, Tokyo
15. August 2010

Plan

Aim

Let A be a **gentle algebra** (these are certain *tame* algebras given by quivers with zero relations of a particular form). Let $\nu_A = \mathbb{L}(\mathrm{Hom}_A(-, A)^*)$ be the **Nakayama functor**

Aim

Let A be a **gentle algebra** (these are certain *tame* algebras given by quivers with zero relations of a particular form). Let $\nu_A = \mathbb{L}(\mathrm{Hom}_A(-, A)^*)$ be the **Nakayama functor** and $\tau_A = \nu_A \circ [-1]$ be the **Auslander-Reiten translation.**

Aim

Let A be a **gentle algebra** (these are certain *tame* algebras given by quivers with zero relations of a particular form). Let $\nu_A = \mathbb{L}(\mathrm{Hom}_A(-, A)^*)$ be the **Nakayama functor** and $\tau_A = \nu_A \circ [-1]$ be the **Auslander-Reiten translation.**

Aim

Describe the Verdier quotient

$$\mathcal{D} = \mathrm{D}^b(A - \mathrm{mod})/\{\tau_A\text{–}\textit{invariant complexes}\}.$$

1. *Is* $\mathcal{D}$ Hom*–finite ?*
2. *What are the indecomposable objects/representation type/AR-quiver.*
3. K*-theory.*
4. *Is* $\mathcal{D}$ *idempotent complete ?*

Example I

Let $A = k(\ 1 \underset{b}{\overset{a}{\rightrightarrows}} 2 \underset{d}{\overset{c}{\rightrightarrows}} 3\ )/(ca, db)$.

Example I

Let $A = k(\, 1 \underset{b}{\overset{a}{\rightrightarrows}} 2 \underset{d}{\overset{c}{\rightrightarrows}} 3 \,)/(ca, db)$.

Question

Can we describe the Verdier quotient

$$\mathcal{D} = \mathrm{D}^b(A - \mathrm{mod})/\{\tau_A\text{–}\textit{invariant complexes}\}?$$

Example I

Let $A = k(\ 1 \overset{a}{\underset{b}{\rightrightarrows}} 2 \overset{c}{\underset{d}{\rightrightarrows}} 3\ )/(ca, db)$.

Question

Can we describe the Verdier quotient

$$\mathcal{D} = \mathrm{D}^b(A - \mathrm{mod})/\{\tau_A\text{–}\textit{invariant complexes}\}?$$

Idea

Find a geometric interpretation of $\mathrm{D}^b(A - \mathrm{mod})$.

The following example shows how this might look like.

Example II

The well known tilting equivalence

$$\mathrm{D}^b(\mathrm{Coh}\,\mathbb{P}^1) \longrightarrow \mathrm{D}^b\left(\mathrm{rep}\left(\bullet \rightrightarrows \bullet \right)\right)$$

Example II

The well known tilting equivalence

$$\mathrm{D}^b(\mathrm{Coh}\,\mathbb{P}^1) \longrightarrow \mathrm{D}^b\left(\mathrm{rep}\left(\bullet \rightrightarrows \bullet\right)\right)$$

induces an equivalence of Verdier-quotient categories

$$\mathrm{D}^b(\mathrm{Coh}\,\mathbb{P}^1)/\mathrm{D}^b(\mathrm{tors}\,\mathbb{P}^1) \longrightarrow \mathrm{D}^b(\mathrm{rep}(\bullet \rightrightarrows \bullet))/\{\tau\text{–invariant complexes}\}.$$

Example II

The well known tilting equivalence

$$\mathrm{D}^b(\mathrm{Coh}\,\mathbb{P}^1) \longrightarrow \mathrm{D}^b\left(\mathrm{rep}\left(\bullet \rightrightarrows \bullet\right)\right)$$

induces an equivalence of Verdier-quotient categories

$$\mathrm{D}^b(\mathrm{Coh}\,\mathbb{P}^1)/\mathrm{D}^b(\mathrm{tors}\,\mathbb{P}^1) \longrightarrow \mathrm{D}^b(\mathrm{rep}(\bullet \rightrightarrows \bullet))/\{\tau\text{–invariant complexes}\}.$$

By a result of Miyachi and Bondal-Orlov

$$\mathrm{D}^b(\mathrm{Coh}\,\mathbb{P}^1)/\mathrm{D}^b(\mathrm{tors}\,\mathbb{P}^1) \cong \mathrm{D}^b(\mathrm{Coh}\,\mathbb{P}^1/\mathrm{tors}\,\mathbb{P}^1).$$

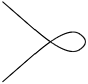
And finally

$$\mathrm{D}^b(\mathrm{Coh}\,\mathbb{P}^1/\mathrm{tors}\,\mathbb{P}^1) \cong \mathrm{D}^b(k(t)\text{–mod}).$$

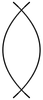
Kodaira cycles

Let $X = E_n$ be a Kodaira cycle of n projective lines. E.g.

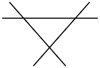
E_1



E_2

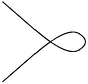


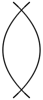
E_3

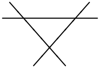


Kodaira cycles

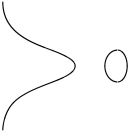
Let $X = E_n$ be a Kodaira cycle of n projective lines. E.g.

E_1 

E_2 

E_3 

They are certain degenerations of elliptic curves



In particular, $\tau_X|_{\mathrm{Perf}(X)} \cong \mathrm{Id}$. Since the varieties X are singular,

$$\mathrm{gl.dim}(\mathrm{Coh}(X)) = \infty.$$

Non-commutative Kodaira cycles

Let $Z = \mathrm{Sing}(X)$ be the singular locus and $\mathcal{I}_Z$ the corresponding ideal sheaf. Set $\mathcal{F} = \mathcal{O}_X \oplus \mathcal{I}_Z$.

Non-commutative Kodaira cycles

Let $Z = \mathrm{Sing}(X)$ be the singular locus and $\mathcal{I}_Z$ the corresponding ideal sheaf. Set $\mathcal{F} = \mathcal{O}_X \oplus \mathcal{I}_Z$. Burban and Drozd equip these curves with an **Auslander sheaf** of $\mathcal{O}_X$-orders

$$\mathcal{A}_X = \mathcal{E}nd_X(\mathcal{F}) \cong \begin{pmatrix} \widetilde{\mathcal{O}} & \mathcal{I} \\ \widetilde{\mathcal{O}} & \mathcal{O} \end{pmatrix}.$$

Non-commutative Kodaira cycles

Let $Z = \mathrm{Sing}(X)$ be the singular locus and $\mathcal{I}_Z$ the corresponding ideal sheaf. Set $\mathcal{F} = \mathcal{O}_X \oplus \mathcal{I}_Z$. Burban and Drozd equip these curves with an **Auslander sheaf** of $\mathcal{O}_X$-orders

$$\mathcal{A}_X = \mathcal{E}nd_X(\mathcal{F}) \cong \begin{pmatrix} \widetilde{\mathcal{O}} & \mathcal{I} \\ \widetilde{\mathcal{O}} & \mathcal{O} \end{pmatrix}.$$

Let $r, s \in X$ be a regular resp. singular point. The completions of the local rings are

$$\widehat{\mathcal{A}}_r \cong \begin{pmatrix} k[\![x]\!] & k[\![x]\!] \\ k[\![x]\!] & k[\![x]\!] \end{pmatrix}, \qquad \widehat{\mathcal{A}}_s \cong \left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3 \right)^{\wedge}.$$

The ringed space $(X, \mathcal{A}_X)$ may be viewed as a non-commutative curve and studied via the category $\mathrm{Coh}(\mathcal{A}_X)$ of coherent $\mathcal{A}_X$-modules.

Remark

$\widehat{\mathcal{A}}_s$ *is the* **Auslander algebra** *of an* A_1*-singularity.*

Categorical resolutions

Since $\operatorname{gl.dim}(\operatorname{QCoh}(\mathcal{A}_X)) = 2$ (local computation), the derived category $\operatorname{D}(\operatorname{QCoh}(\mathcal{A}_X))$ is *smooth* in the sense of Kontsevich and a *categorical resolution* in the sense of Lunts.

Categorical resolutions

Since $\mathrm{gl.dim}(\mathrm{QCoh}(\mathcal{A}_X)) = 2$ (local computation), the derived category $\mathrm{D}(\mathrm{QCoh}(\mathcal{A}_X))$ is *smooth* in the sense of Kontsevich and a *categorical resolution* in the sense of Lunts. More precisely

Theorem (Burban-Drozd '09)

The category $\mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X))$ *has a* **Serre functor** $\mathbb{S}_X$*; thus* $\tau_X = \mathbb{S}_X \circ [-1]$ *is the* **Auslander-Reiten translation**. *There is a fully faithful functor*

$$\mathbb{F} = \mathcal{F} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} - : \mathrm{Perf}(X) \hookrightarrow \mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X))$$

and $\mathrm{im}(\mathbb{F}) = \{\tau_X$ *– invariant complexes*$\} \subset \mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X))$.

Singularity categories

Singularity categories were introduced by Buchweitz '87 and thereafter studied extensively by representation-theorists. In '03 Orlov started to study a 'global' version of them.

Definition (Buchweitz, Orlov)

Let X be a noetherian scheme. The **triangulated category of singularities** is the Verdier-quotient

$$\mathrm{D}_{\mathrm{sg}}(X) = \mathrm{D}^b(\mathrm{Coh}(X))/\mathrm{Perf}(X)$$

Singularity categories

Singularity categories were introduced by Buchweitz '87 and thereafter studied extensively by representation-theorists. In '03 Orlov started to study a 'global' version of them.

Definition (Buchweitz, Orlov)

Let X be a noetherian scheme. The **triangulated category of singularities** is the Verdier-quotient

$$\mathrm{D}_{\mathrm{sg}}(X) = (\mathrm{D}^b(\mathrm{Coh}(X))/\mathrm{Perf}(X))^{\omega}$$

Singularity categories

Singularity categories were introduced by Buchweitz '87 and thereafter studied extensively by representation-theorists. In '03 Orlov started to study a 'global' version of them.

Definition (Buchweitz, Orlov)

Let X be a noetherian scheme. The **triangulated category of singularities** is the Verdier-quotient

$$\mathrm{D}_{\mathrm{sg}}(X) = (\mathrm{D}^b(\mathrm{Coh}(X))/\mathrm{Perf}(X))^{\omega}$$

The following theorem due to Orlov shows that $\mathrm{D}_{\mathrm{sg}}(X)$ may be understood locally.

Theorem (Orlov)

Let X be quasi-projective with isolated singularities. There is a **block decomposition**

$$\mathrm{D}_{\mathrm{sg}}(X) \cong \prod_{x \in \mathrm{Sing}(X)} \mathrm{D}^b(\widehat{\mathcal{O}}_x - \mathrm{mod})/\mathrm{K}^b(\mathrm{proj}\, \widehat{\mathcal{O}}_x).$$

Non-commutative singularity categories

The following proposition is a non-commutative analogue of Orlov's Theorem.

Proposition

Let $X = E_n$ *be a Kodaira cycle and* $\{s_1, \cdots, s_n\} = \mathrm{Sing}(X)$.

$$\Big(\mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X))/\mathrm{Perf}(X)\Big)^{\omega} \cong \prod_{i=1}^{n} \mathrm{D}^b(\widehat{\mathcal{A}}_{s_i} - \mathrm{mod})/\mathrm{K}^b(\mathrm{add}\, P_2),$$

Non-commutative singularity categories

The following proposition is a non-commutative analogue of Orlov's Theorem.

Proposition

Let $X = E_n$ *be a Kodaira cycle and* $\{s_1, \cdots, s_n\} = \mathrm{Sing}(X)$.

$$\Big(\mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X))/\mathrm{Perf}(X)\Big)^{\omega} \cong \prod_{i=1}^{n} \mathrm{D}^b(\widehat{\mathcal{A}}_{s_i} - \mathrm{mod})/\mathrm{K}^b(\mathrm{add}\, P_2),$$

where

$$\widehat{\mathcal{A}}_{s_i} \cong \left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3 \right)^{\wedge} \cong \begin{pmatrix} k[\![x]\!] \times k[\![y]\!] & (x, y) \\ k[\![x]\!] \times k[\![y]\!] & k[\![x, y]\!]/xy \end{pmatrix}$$

We call the LHS **non-commutative singularity category** $\mathrm{D}_{\mathrm{nc-sg}}(X)$ *and the blocks on the right* **nodal blocks** $\mathrm{D}_{\mathrm{nodal}}$.

Tilting

Theorem (Burban-Drozd '09)

The category $\mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X))$ *admits a tilting complex* $H^\bullet$*. Let* $\Lambda_X = \mathrm{End}(H^\bullet)$*. Thus we have an equivalence of triangulated categories*

$$\mathbb{R}\mathrm{Hom}(H^\bullet, -) : \mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X)) \longrightarrow \mathrm{D}^b(\Lambda_X - \mathrm{mod})$$

Tilting

Theorem (Burban-Drozd '09)

The category $\mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X))$ *admits a tilting complex* $H^\bullet$. *Let* $\Lambda_X = \mathrm{End}(H^\bullet)$. *Thus we have an equivalence of triangulated categories*

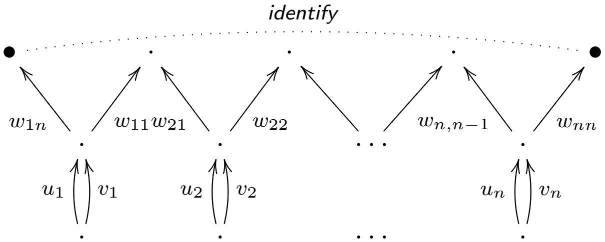
$$\mathbb{R}\mathrm{Hom}(H^\bullet, -) : \mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X)) \longrightarrow \mathrm{D}^b(\Lambda_X - \mathrm{mod})$$

inducing an exact equivalence

$$\Big(\mathrm{D}^b(\mathrm{Coh}(\mathcal{A}_X))/\mathrm{Perf}(X)\Big)^\omega \longrightarrow \Big(\mathrm{D}^b(\Lambda_X - \mathrm{mod})/\{\tau - \textit{invariant cpxs}\}\Big)^\omega$$

Gentle algebras

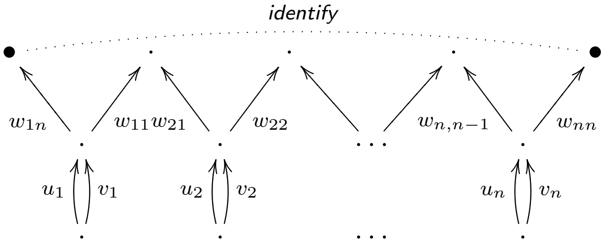
Let $X = E_n$ be a Kodaira cycle of n projective lines. $\Lambda_X = \mathrm{End}(\mathrm{H}^\bullet)$ is a **gentle algebra** given by the following quiver with relations.



$$w_{ii}v_i = 0 \quad 1 \leq i \leq n;$$
$$w_{i,i-1}u_i = 0 \quad 1 < i \leq n;$$
$$w_{1n}u_1 = 0.$$

Gentle algebras

Let $X = E_n$ be a Kodaira cycle of n projective lines. $\Lambda_X = \mathrm{End}(\mathrm{H}^\bullet)$ is a **gentle algebra** given by the following quiver with relations.



$$\begin{array}{ll} w_{ii}v_i = 0 & 1 \leq i \leq n; \\ w_{i,i-1}u_i = 0 & 1 < i \leq n; \\ w_{1n}u_1 = 0. & \end{array}$$

In particular, for $n = 1$ we obtain

$$\Lambda_X = A = k\left(1 \underset{b}{\overset{a}{\rightrightarrows}} 2 \underset{d}{\overset{c}{\rightrightarrows}} 3 \right)/(ca, db).$$

Description of the nodal block I

Let us describe the Verdier quotient

$$\mathrm{D}_{\mathrm{nodal}} = \mathrm{D}^b\left(\left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3\right)^{\wedge}\right)/\mathrm{K}^b(\mathrm{add}P_2).$$

Description of the nodal block I

Let us describe the Verdier quotient

$$\mathrm{D}_{\mathrm{nodal}} = \mathrm{D}^b\left(\left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3\right)^{\wedge}\right)/\mathrm{K}^b(\mathrm{add}P_2).$$

Let $i, j \in \{1, 3\}$. Denote by $\mathcal{S}_i(l)$ the complex

$$P_i \longrightarrow P_2 \longrightarrow P_2 \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_j$$

with l non-vanishing terms, P_i in degree 0, and differentials given by minimal non-trivial paths. These are called **minimal strings.**

Description of the nodal block I

Let us describe the Verdier quotient

$$\mathrm{D}_{\mathrm{nodal}} = \mathrm{D}^b\left(\left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3\right)^{\wedge}\right)/\mathrm{K}^b(\mathrm{add}P_2).$$

Let $i, j \in \{1, 3\}$. Denote by $\mathcal{S}_i(l)$ the complex

$$P_i \longrightarrow P_2 \longrightarrow P_2 \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_j$$

with l non-vanishing terms, P_i in degree 0, and differentials given by minimal non-trivial paths. These are called **minimal strings.**

Example

$$\mathcal{S}_1(3) = \cdots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{\cdot b} P_2 \xrightarrow{\cdot d} P_3 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\mathcal{S}_3(4) = \cdots \longrightarrow 0 \longrightarrow P_3 \xrightarrow{\cdot c} P_2 \xrightarrow{\cdot ab} P_2 \xrightarrow{\cdot d} P_3 \longrightarrow 0 \longrightarrow \cdots$$

Description of the nodal block II

Theorem (K.)

Let $i \neq j \in \{1, 3\}$*,* $n \in \mathbb{Z}$ *and* $l \in \mathbb{N}_{\geq 3}$*.*

1. $\mathrm{D}_{\mathrm{nodal}}$ *is* **representation–discrete**. *More precisely, the set of isomorphism classes of indecomposable objects consists of two* $\mathbb{Z}$*–families* $P_1[n]$ *and* $P_3[n]$ *and two* $\mathbb{N} \times \mathbb{Z}$*-families* $\mathcal{S}_1(l)[n]$ *and* $\mathcal{S}_3(l)[n]$*.*

Description of the nodal block II

Theorem (K.)

Let $i \neq j \in \{1,3\}$, $n \in \mathbb{Z}$ *and* $l \in \mathbb{N}_{\geq 3}$.

1. $\mathrm{D}_{\mathrm{nodal}}$ *is* **representation–discrete**. *More precisely, the set of isomorphism classes of indecomposable objects consists of two* $\mathbb{Z}$*–families* $P_1[n]$ *and* $P_3[n]$ *and two* $\mathbb{N} \times \mathbb{Z}$*-families* $\mathcal{S}_1(l)[n]$ *and* $\mathcal{S}_3(l)[n]$.
2. $\mathrm{D}_{\mathrm{nodal}}$ *is* **Hom–finite**. *Let* X *and* Y *be indecomposable objects in* $\mathrm{D}_{\mathrm{nodal}}$ *then*

$$\dim_k \mathrm{Hom}_{\mathrm{D}_{\mathrm{nodal}}}(X, Y) \leq 1.$$

Description of the nodal block II

Theorem (K.)

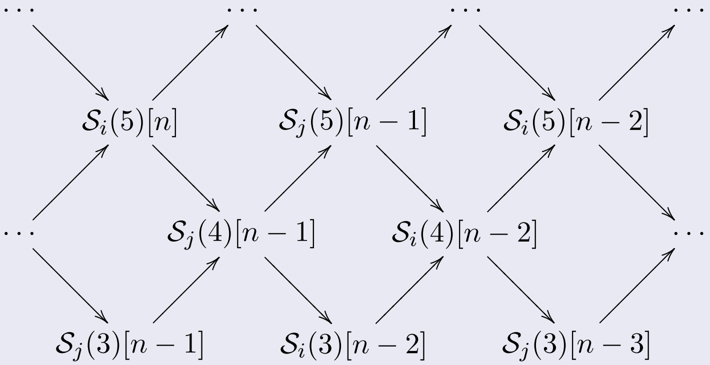
Let $i \neq j \in \{1, 3\}$, $n \in \mathbb{Z}$ *and* $l \in \mathbb{N}_{\geq 3}$.

1. $\mathrm{D}_{\mathrm{nodal}}$ *is* **representation–discrete**. *More precisely, the set of isomorphism classes of indecomposable objects consists of two* $\mathbb{Z}$*–families* $P_1[n]$ *and* $P_3[n]$ *and two* $\mathbb{N} \times \mathbb{Z}$*-families* $\mathcal{S}_1(l)[n]$ *and* $\mathcal{S}_3(l)[n]$.
2. $\mathrm{D}_{\mathrm{nodal}}$ *is* **Hom–finite**. *Let* X *and* Y *be indecomposable objects in* $\mathrm{D}_{\mathrm{nodal}}$ *then*
$$\dim_k \mathrm{Hom}_{\mathrm{D}_{\mathrm{nodal}}}(X, Y) \leq 1.$$
3. *The* **Auslander–Reiten quiver** *of* $\mathrm{D}_{\mathrm{nodal}}$ *consists of two* $\mathbb{A}_{\infty}^{\infty}$*–components*
$$\cdots \longrightarrow P_i[n] \longrightarrow P_j[n-1] \rightarrow P_i[n-2] \rightarrow P_j[n-3] \longrightarrow \cdots$$

Description of the nodal block III

Theorem (K.)

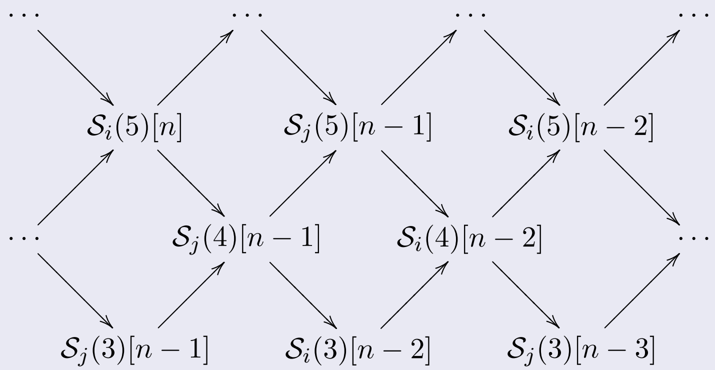
and two $\mathbb{Z}\mathbb{A}_\infty$–components.



Description of the nodal block III

Theorem (K.)

and two $\mathbb{Z}\mathbb{A}_\infty$–components.



4. $\mathrm{K}_0(\mathrm{D}_{\mathrm{nodal}}) \cong \mathbb{Z}^2$ *(using negative* K*-theory.)*

Description of the nodal block IV

Theorem (K.)

5. *Let* $O = k[\![x, y]\!]/(xy)$ *and*

$$\widehat{\mathcal{A}}_s = \left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3 \right)^{\wedge}.$$

Description of the nodal block IV

Theorem (K.)

5. *Let* $O = k[\![x,y]\!]/(xy)$ *and*

$$\widehat{\mathcal{A}}_s = \left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3 \right)^{\wedge}.$$

We have a commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{D}^b(\widehat{\mathcal{A}}_s - \mathrm{mod}) & \xrightarrow{\mathbb{R}\mathrm{Hom}(P_2,-)} & \mathrm{D}^b(O - \mathrm{mod}) \\
 & & \downarrow{\scriptstyle \textit{can.}} & & \downarrow{\scriptstyle \textit{can.}} \\
\mathrm{add}\,(S_i(l)[n]) & \xrightarrow{\textit{incl.}} & \mathrm{D}_{\mathrm{nodal}} & \xrightarrow{\textit{induced}} & \underline{\mathrm{CM}}(O)
\end{array}$$

Description of the nodal block IV

Theorem (K.)

5. *Let* $O = k[\![x,y]\!]/(xy)$ *and*

$$\widehat{\mathcal{A}}_s = \left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{d}{\overset{c}{\rightleftarrows}} 3 \right)^{\wedge}.$$

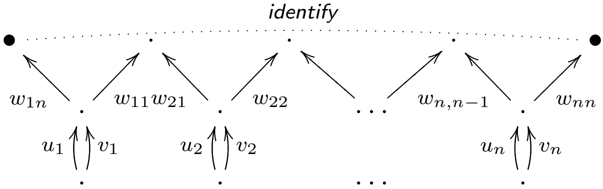
We have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{D}^b(\widehat{\mathcal{A}}_s - \mathrm{mod}) & \xrightarrow{\mathbb{R}\mathrm{Hom}(P_2,-)} & \mathrm{D}^b(O - \mathrm{mod}) \\ & & \downarrow \text{can.} & & \downarrow \text{can.} \\ \mathrm{add}\,(S_i(l)[n]) & \xrightarrow{\text{incl.}} & \mathrm{D}_{\mathrm{nodal}} & \xrightarrow{\text{induced}} & \underline{\mathrm{CM}}(O) \end{array}$$

relating our non-commutative singularity category $\mathrm{D}_{\mathrm{nc-sg}}(X)$ *to the known (commutative) singularity category* $\mathrm{D}_{\mathrm{sg}}(\mathrm{X})$*.*

Summary

Let Λ_X be the gentle algebra given by



$$\begin{array}{ll} w_{ii}v_i=0 & 1\leq i\leq n; \\ w_{i,i-1}u_i=0 & 1<i\leq n; \\ w_{1n}u_1=0. & \end{array}$$

and $\mathcal{D} = \left(\mathrm{D}^b(A - \mathrm{mod})/\{\tau_A\text{–invariant complexes}\}\right)^{\omega}$.

1. $\mathcal{D} \cong \mathrm{D}_{\mathrm{nodal}} \oplus \cdots \oplus \mathrm{D}_{\mathrm{nodal}}$ (**block-decomposition**).
2. $\mathcal{D}$ is **Hom–finite**.
3. $\mathcal{D}$ is **representation-discrete**.
4. $\mathrm{K}_0(\mathcal{D}) \cong (\mathbb{Z}^2)^{\oplus n}$.