Matrix Factorisations:

Knörrer's periodicity and beyond.

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Klein – Gordon equation:

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<u>But</u>: Does <u>not</u> agree with experiments!

Aim (Dirac): Factor Klein - Gordon eq. as follows

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But No polynomial r satisfies $r^2 = f$.

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and a Nobel prize for Dirac in 1933.

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$$A \cdot B = f \cdot \mathrm{Id}_m = B \cdot A$$

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$$\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \right)$$

"(direct) sum" of MFs

is a MFs of f.

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In other words, (A, A^{adj}) is a MF for $\det A$.

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Thm (Auslander–Buchsbaum–Eisenbud–Serre): $\underline{MF}(f) = 0 \iff \{f=0\}$ is a smooth variety

Knörrer's periodicity

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$$\rightsquigarrow \left(\begin{pmatrix} -A & (y+iz)\,\mathrm{Id} \\ (y-iz)\,\mathrm{Id} & B \end{pmatrix}, \begin{pmatrix} -B & (y+iz)\,\mathrm{Id} \\ (y-iz)\,\mathrm{Id} & A \end{pmatrix} \right) \quad \text{MF of } g = f + y^2 + z^2$$

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Thm (Knörrer 1987) $\underline{\mathrm{MF}}(f) \cong \underline{\mathrm{MF}}(f + y^2 + z^2)$

"up to sums of trivial MFs, there is a bijection between MFs of f and $f + y^2 + z^2$."

Rem: In general $\underline{MF}(f) \neq \underline{MF}(f+y^2)$!

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In other words the sequence

$$\underline{MF}(f),\ \underline{MF}(f+y_1^2),\ \underline{MF}(f+y_1^2+y_2^2),\ \underline{MF}(f+y_1^2+y_2^2+y_3^2),\ldots$$

is 2-periodic. This is called Knörrer's periodicity.

... and beyond.

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In particular: Singularity categories generalise $\underline{MF}(f)$!

and Knörrer's theorem translates to

$$\mathcal{D}_{Sg}\left(\frac{\mathbb{C}[x_0,\dots,x_d,y,z]}{(f+y^2+z^2)}\right) \cong \mathcal{D}_{Sg}\left(\frac{\mathbb{C}[x_0,\dots,x_d]}{(f)}\right)$$

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<u>Q</u>: Can this be extended beyond hypersurfaces

$\mathbb{C}[x_0,\dots,x_d]/(f)$?

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$$G := \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle \subset GL(2, \mathbb{C})$$

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Rem: Proof uses (non-commutative) resolutions of singularities

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$\underline{a=1}$: Dong Yang $\Longrightarrow$ $\mathcal{D}_{sg}\left(\dfrac{\mathbb{C}[x_0,\dots,x_r]}{(x_ix_j-x_kx_l \mid i+j=k+l)}\right) \cong \mathcal{D}_{sg}\left(\dfrac{\mathbb{C}[x_1,\dots,x_{r-1}]}{(x_1,\dots,x_{r-1})^2}\right)$

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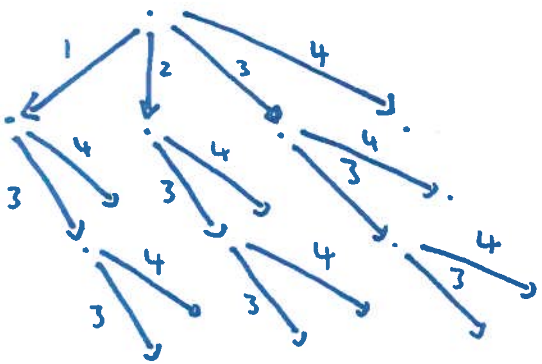

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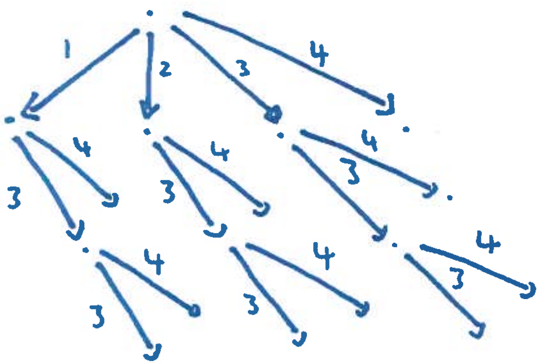

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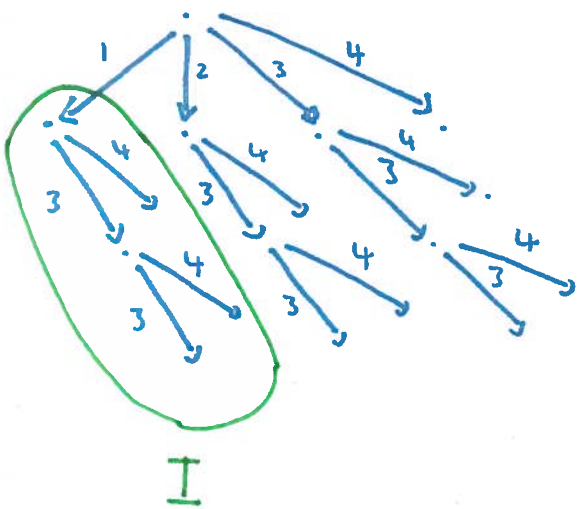

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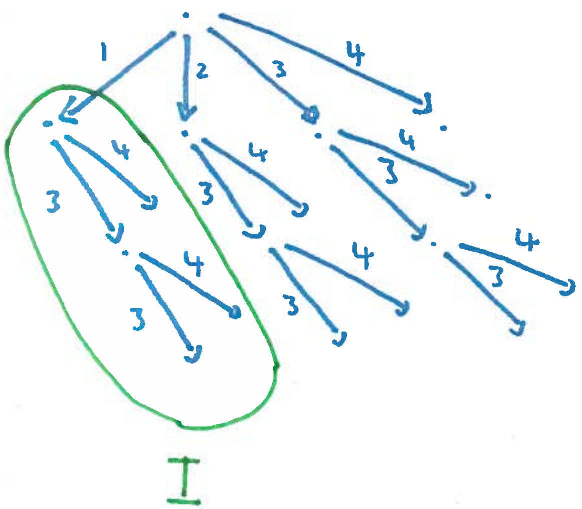

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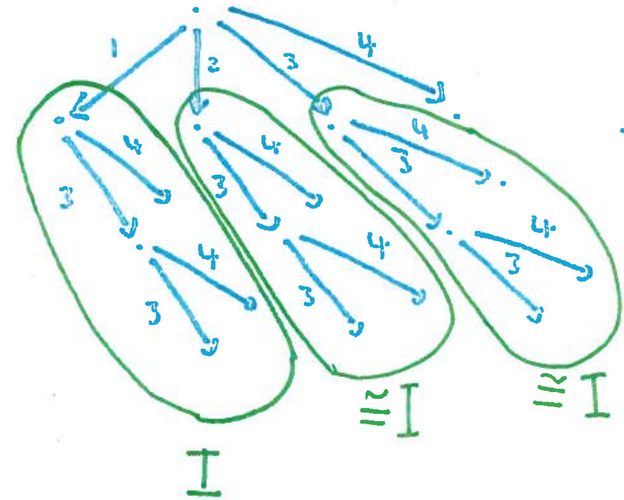

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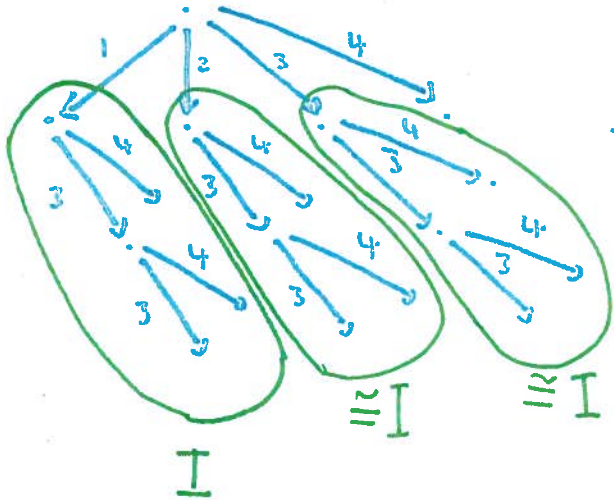

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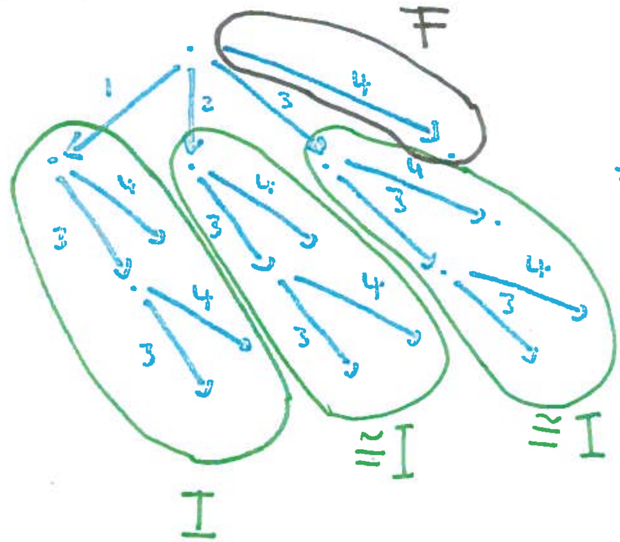

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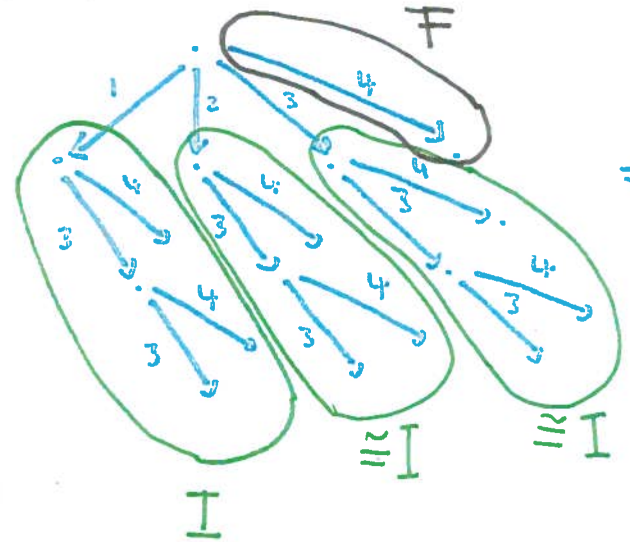

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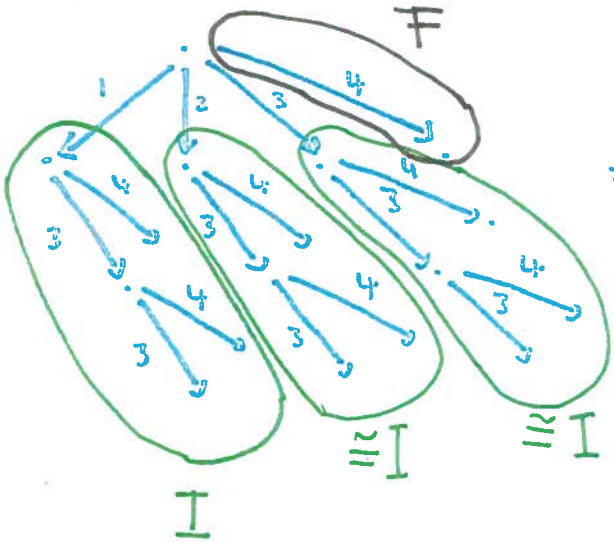

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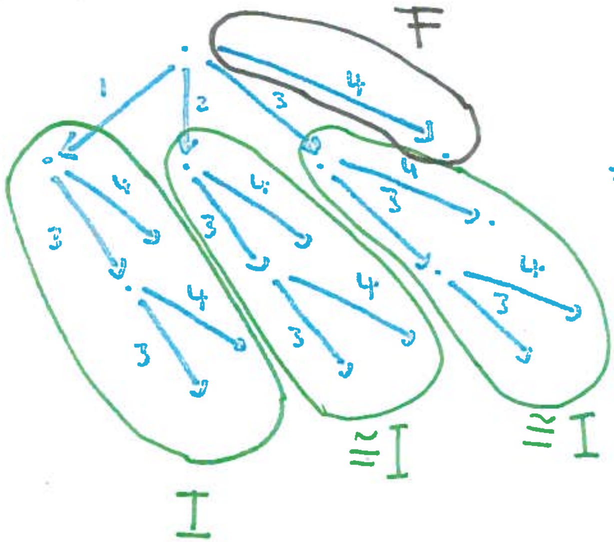

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Euclidean algorithm

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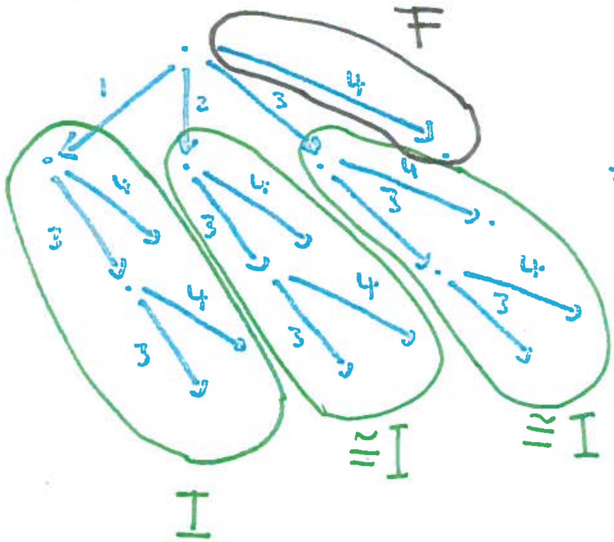

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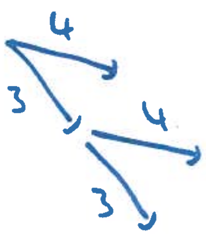

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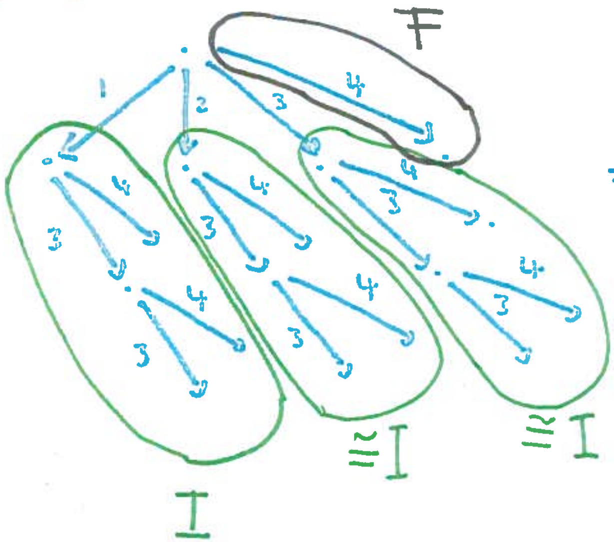

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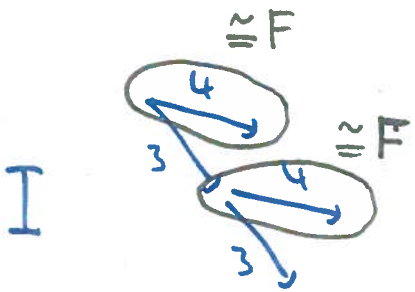

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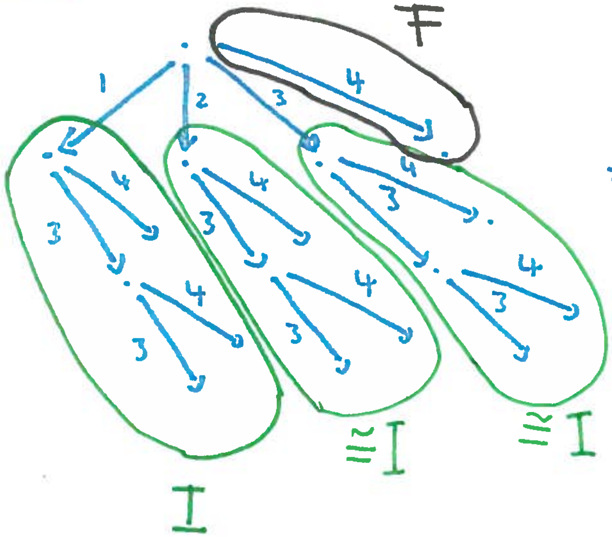

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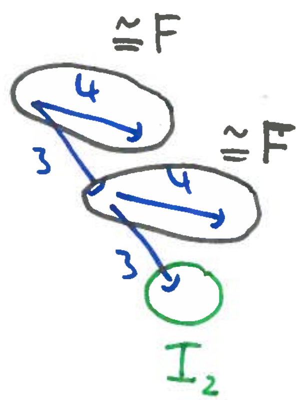

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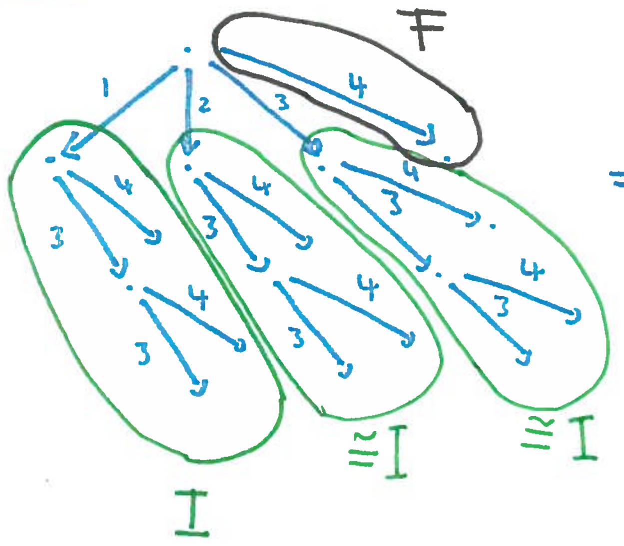

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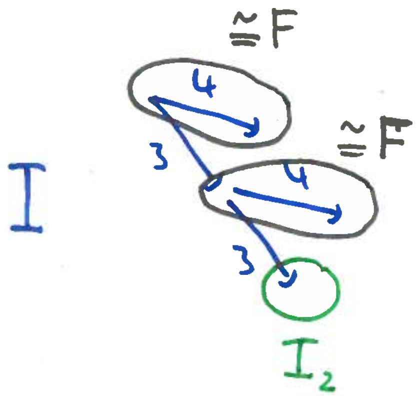

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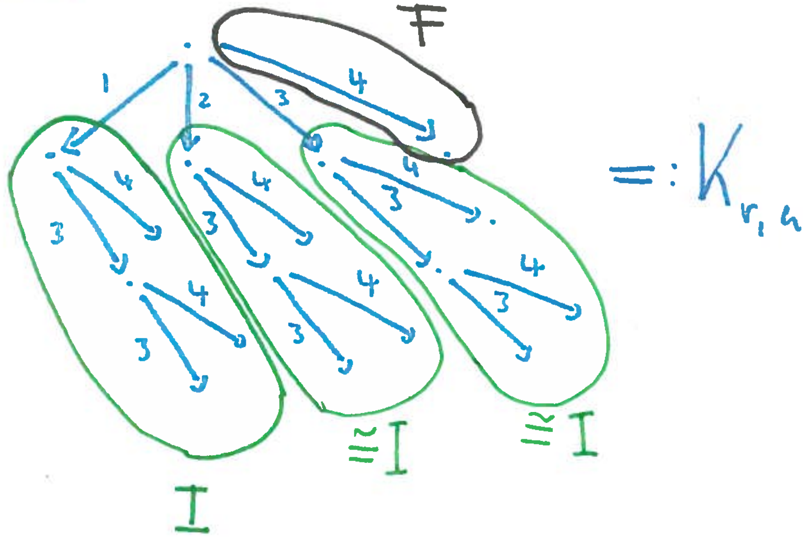

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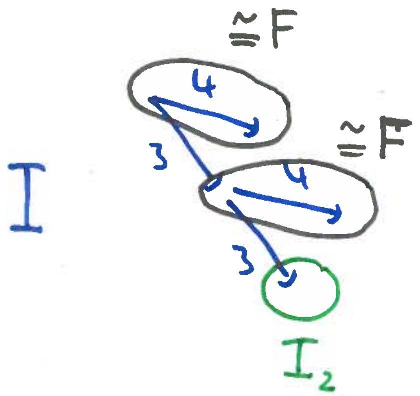

$$[I : F] = 2$$

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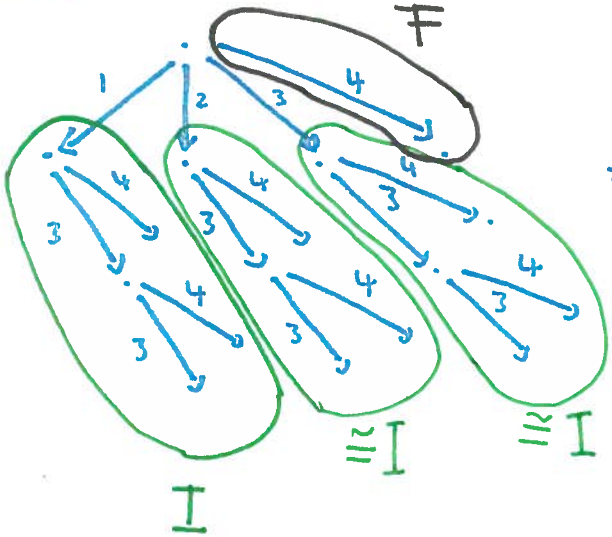

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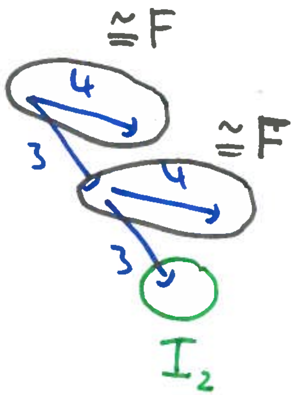

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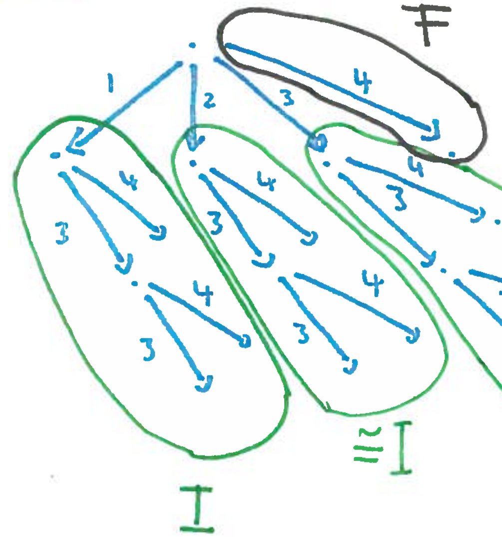

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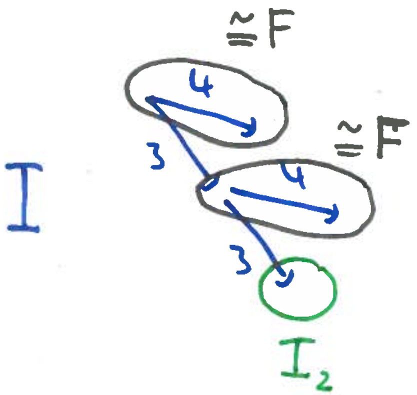

$$\left.\begin{array}{l} [I : F] = 2 \\ \dim I_2 = 1 \end{array}\right\}$$

$$\dim I = [I : F] \cdot \dim F + \dim I_2$$
$$5 = 2 \cdot 2 + 1$$