Relative singularity categories: dg-models & applications

jt. w. D. Yang

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1. Introduction & Motivation

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Define $$\Delta_e(A) = \frac{D^b(A\text{-mod})}{\mathrm{thick}(Ae)}$$

Singularity category of A relative to e

Often $A = \mathrm{End}_R(R \oplus M)$, R noeth., $M \in \text{mod-}R$

$e = R \oplus M \twoheadrightarrow R \hookrightarrow R \oplus M$

$$\Delta_e(A) = \frac{D^b(A)}{\mathrm{thick}(Ae)}$$

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<u>Rem</u>: $D_{sg}(A) = \Delta_1(A)$

<u>Idea</u>: Use $\Delta_e(A)$ to gain insight into structure of $D_{sg}(eAe)$

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Rem: Overlaps with Amiot-Iyama-Reiten who use different techniques.

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<u>Building on</u>: Keller – Hua '18

Booth '18/'19

Hanhoffer – Van den Bergh

K. – Yang '16/'18

2) Karmazyn – Wemyss (announced '19)

Proof of Donovan – Wemyss Conjecture in Homological Minimal Model Program

⌜Also recovers one of the main results of K. – Yang: "Relative singularity categories I".⌟

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cyclic quotient surface singularity

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generalising earlier works of Hille-Perling,

Kawamata and Kuznetsov.

And inspiring further works of Kawamata '19 and K.-Pavic-Shinder '19

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4) Inspired Morita-type theorem for Frobenius exact categories

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Which was applied to:

a) Describe Iyama-Wemyss's triangulated category for rational surface singularities

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(Iyama - K. - Wemyss - Yang)

b) Categorify certain cluster algebras

(Pressland '15 building on Jensen-King-Su)

(see also Keller - Scherotzke)

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* * *

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Cline-Parshall-Scott $\Longrightarrow$ $\Delta_e(A) \cong D^b(A/AeA)$

and $D_{sg}(eAe) = 0$ boring!

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indeed $D^b(k)$ has only trivial Δ-ted subcategories

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Building on work of Keller, Nicolas–Saorin, ...

we can show the following.

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where $D_{fg}(B) \subset D(B)$ is the subcat. of dg-modules M s.th. $H^*(M)$ is fin. gen. over A/AeA.

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This is a gen. cluster category (Amiot) if B is a Ginzburg dg-algebra $\Gamma(Q, W)$

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<u>Thm</u> (K-Yang '18)

R (non-complete) 3 dim. Gor. quotient singularity or 3 dim toric singularity. (possibly non-isolated sing)

Then $D_{sg}(R) \cong \mathcal{C}^s(Q,W)$

with explicit Q and W.

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<u>Preparation</u>:

<u>Def</u>: Q graded quiver with finitely many vertices. Call dg algebras (kQ, d) <u>dg quiver algebras</u>.

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- Assume moreover that $(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) \neq I$ for all i

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These are all arrows of $\tilde{Q}$.

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<u>Example:</u>

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__Example:__

$$A = \frac{k\left(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}\right)}{(ab)}, \quad \text{then}$$

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Theorem (K.–Yang '18)

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Cor: $\mathrm{per}\left({}^{\tilde{A}}/_{\tilde{A}e\tilde{A}}\right) \Big/ D_{fg}\left({}^{\tilde{A}}/_{\tilde{A}\tilde{e}\tilde{A}}\right) \cong D_{sg}(eAe)$

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$$\rightsquigarrow \quad B = \tilde{A} / \tilde{A} e_2 \tilde{A} = k[s(ab)], \; d = 0$$

$$\frac{\mathrm{per}(B)}{D_{fd}(B)} \cong D_{sg}\left(\frac{k[x]}{(x^2)} \right)$$

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$$\rightsquigarrow B = \tilde{A} / \tilde{A} e_2 \tilde{A} = k[s(ab)], \quad d = 0$$

$$\mathrm{per}(k[t, t^{-1}]) \simeq \frac{\mathrm{per}(B)}{D_{fd}(B)} \simeq D_{sg}\left(\frac{k[x]}{(x^2)}\right)$$

$\deg(t) = -1$, $d = 0$

Example: $R = k[x]/(x^2)$, $A = \mathrm{End}_R(R \oplus k)$

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Lemma (Keller) $\Rightarrow$ $\tilde{A} = \left(k \left(\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[loop right, dashed, "\delta(ab)"] \end{tikzcd} \right), d(\delta(ab)) = ab \right)$

$\rightsquigarrow B = \tilde{A}/\tilde{A}e_2\tilde{A} = k[\delta(ab)]$, $d = 0$

$$\mathrm{per}(k[t,t^{-1}]) \cong \frac{\mathrm{per}(B)}{D_{fd}(B)} \cong D_{sg}\left(\frac{k[x]}{(x^2)}\right)$$

$\deg(t) = -1$, $d = 0$

e.g. via universal loc. and work of Schofield, Neeman–Ranicki, Neeman...

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Leavitt path algebra of Q^{op}

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Set $\mathcal{O}^\lambda := \mathcal{O}^\lambda(\tilde{Q}) := e_0 \Pi^\lambda(\tilde{Q}) e_0$

(non-comm.) deformation of Kleinian sing. $\mathcal{O}^0$

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Kleinian sing. of type Dynkin-$Q^{(i)}$

where $Q^{(1)} \sqcup \dots \sqcup Q^{(r)} = \widetilde{Q} \setminus \{\text{vertices } i \text{ with } \lambda_i \neq 0\}$