Describing Leclerc's Frobenius categories as categories of Gorenstein projective modules

cf. arXiv 1709.04785

Clusters & braids seminar

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1. Motivation and background

Setup: G simple Lie-group $/\mathbb{C}$ of Dynkin type $Q = A_n, D_n$ or $E_{6,7,8}$

(e.g. $G = SL_{n+1}(\mathbb{C})$ for $Q = A_n$)

with Borel subgroup $B \subset G$

(e.g. $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \subset SL_{n+1}(\mathbb{C})$)

and Borel subgroup $B_- \subset G$ opposite to B

(e.g. $B_- = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \cdots & * \end{pmatrix} \subset SL_{n+1}(\mathbb{C})$)

and Weyl group W

(e.g. $W = S_{n+1}$ given by permutation matrices)

$$G/B = \bigsqcup_{w \in W} BwB/B =: \bigsqcup_{w \in W} C_w$$

flag variety

Schubert decompos.

Schubert cell

$$\left(\text{e.g. } C_{w_0} = \frac{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}}{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}} = \left\{ \begin{pmatrix} * & \cdots & * & 1 \\ \vdots & & \iddots & \\ * & \iddots & & \\ 1 & & & 0 \end{pmatrix} B \right\} \cong \mathbb{C}^{\binom{n+1}{2}} \cong \mathbb{C}^{\ell(w_0)} \right.$$

$$\left. - \text{ in general } C_w \cong \mathbb{C}^{\ell(w)} \right)$$

$$G/B = \bigsqcup_{w \in W} B_- wB/B =: \bigsqcup_{w \in W} C^w$$

opposite Schubert cell

$$\left(\text{e.g. } C^{w_0} = \begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \cdots & * \end{pmatrix}\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \Big/ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} = \left\{ \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} B \right\} \cong \{pt\} \cong \mathbb{C}^0 \right.$$

$$\left. - \text{ in general } C^w \cong \mathbb{C}^{\ell(w_0) - \ell(w)} \right)$$

$$G/B = \bigsqcup_{v,w \in W} C^v \cap C_w =: \bigsqcup_{v,w \in W} \mathring{\mathcal{R}}_{v,w}$$

(open) Richardson variety

(e.g. $\mathring{\mathcal{R}}_{w_0,w_0} = \left\{ \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} B \right\} \cong \{pt\}$

- more generally, $\mathring{\mathcal{R}}_{v,w}$ is smooth, irreducible, affine variety of dim. $\ell(w)-\ell(v)$
(and $\mathring{\mathcal{R}}_{v,w} \neq \emptyset$ iff $v \leq w$))

<u>Thm</u> [GLS] The coordinate rings of C_w & C^w admit a cluster algebra structure.

<u>Pf</u>: Uses "categorification".

More precisely, for each $w \in W$

there are subcategories

$\mathcal{C}_w, \mathcal{C}^w \subset \operatorname{mod} \underbrace{\Pi(Q(G))}$

preprojective algebra of Dynkin type $Q(G)$

with a "cluster structure" [BIRSc].

In particular, $\mathcal{C}_w, \mathcal{C}^w$ contain

cluster-tilting objects

$T = P \oplus M$, with P projective gen.

$\rightsquigarrow \Gamma_T :=$ quiver of $\operatorname{End}(T)$

is part of a seed of cluster alg.

Leclerc suggests an analogous approach for $\mathring{R}_{v,w} = C^v \cap C_w$ using categories $\mathcal{C}_{v,w} := \mathcal{C}^v \cap \mathcal{C}_w$.

Very recently: completed & generalized by Casals, $(\text{Gorsky})^2$, Le, Shen & Simental

(cf. also Galashin, Lam, Sherman-Bennett & Speyer

Rem: In categorification, work with

$$\Gamma_T = \text{quiver of } \mathrm{End}_{\mathcal{C}_{v,w}}(T)$$

cluster tilting object

projective generator of $\mathcal{C}_{v,w}$. Corresponds to "frozen variables" in cluster algebra

$$T \cong P_{v,w} \oplus M$$

Today: More interested in

$$\mathrm{End}_{\mathcal{C}_{v,w}}(P_{v,w})$$

2. From pairs of torsion pairs to Frobenius categories

Goal:

Explain why Leclerc's category

$$\mathcal{C}_{v,w} := \mathcal{C}^v \cap \mathcal{C}_w \subset \operatorname{mod}\Pi$$

is Frobenius.

<u>Def</u>: Let $\mathcal{A}$ be abelian category.

A torsion pair consists of

full subcategories

$\mathcal{T} \subset \mathcal{A}$ torsion class

$\mathcal{F} \subset \mathcal{A}$ torsion free class

satisfying

(T1) $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$

(T2) For all $X \in \mathcal{A}$ there exists

$$0 \to \underset{\mathcal{T}}{\underset{\rotatebox{90}{$\in$}}{t(X)}} \longrightarrow X \longrightarrow \underbrace{X/t(X)}_{=: f(X) \in \mathcal{F}} \longrightarrow 0 \quad \text{ex}$$

Examples:

(a) $\mathcal{A} = \mathrm{Ab}$ cat. of abelian groups

$\mathcal{F} =$ torsion-free groups

$\mathcal{T} =$ torsion groups

(b) $\mathcal{A} = \text{mod-}\Lambda$, Λ f.dim. algebra

$$\begin{cases} \mathcal{T}_i = \operatorname{add} S_i, & S_i \text{ simple } \Lambda\text{-module} \\ \mathcal{F}_i = \{M \mid \operatorname{Hom}_\Lambda(S_i, M) = 0\} = \{M \mid \operatorname{soc}_i M = 0\} \end{cases}$$

(c) If $\Lambda = \Pi$ preprojective algebra of Dynkin type

then "iterating" the definition of $\mathcal{F}_i$ yields ALL torsion-free $\mathcal{F} \subset \operatorname{mod} \Pi$

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Remarks:

(a) (T1), (T2) and universal properties of kernel & cokernel $\Rightarrow$ t and f define functors $\left\{ \begin{array}{l} t : \mathcal{A} \longrightarrow \mathcal{T} \\ f : \mathcal{A} \longrightarrow \mathcal{F} \end{array} \right\}$

(b) $\operatorname{Hom}(X, \mathcal{F}) = 0 \iff X \in \mathcal{T}$

Hence, $\mathcal{T}$ is closed under extensions & quotients

(c) $\operatorname{Hom}(\mathcal{T}, Y) = 0 \iff Y \in \mathcal{F}$

Hence, $\mathcal{F}$ is closed under extensions & subobjects

(d) $t(X) \longrightarrow X$ is a left $\mathcal{T}$-approximation. Indeed:

$$
\begin{tikzcd}
0 \arrow[r] & t(X) \arrow[r] & X \arrow[r] & f(X) \arrow[r] & 0 \\
 & & T \arrow[u] \arrow[ul, dotted, "\exists"] \arrow[ur, "0"'] & &
\end{tikzcd}
$$

Dually,

$$X \longrightarrow f(X) \quad \text{is a}$$

right $\mathcal{F}$-approximation.

Prop: Let $\mathcal{A}$ be an abelian category with torsion pairs $(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2)$.

Then $\mathcal{C}_{12} := \mathcal{T}_1 \cap \mathcal{F}_2$ satisfies

(a)
$$\mathcal{T}_1 \xrightarrow{f_2} \mathcal{C}_{12} \xleftarrow{t_1} \mathcal{F}_2$$

(b) $\mathcal{C}_{12}$ is closed under extensions in $\mathcal{A}$.
Hence, $\mathcal{C}_{12}$ inherits exact structure of $\mathcal{A}$.

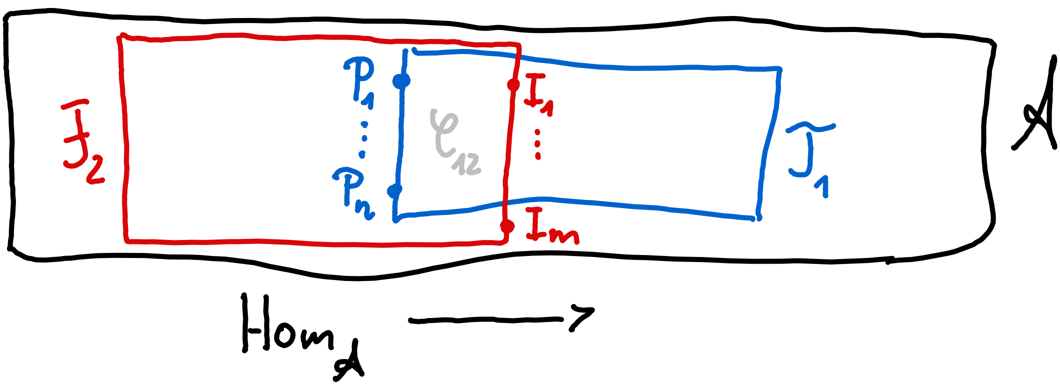
(c) $\mathcal{C}_{12}$ has kernels & cokernels, e.g.

$$\underbrace{t_1(\ker_{\mathcal{A}} f)}_{=: \ker_{\mathcal{C}_{12}} f} \hookrightarrow \ker_{\mathcal{A}} f \hookrightarrow C \xrightarrow{f} C'$$

is a kernel of f.

Hence, $\mathcal{C}_{12}$ is preabelian category

"Morally" the situation look like:



(d) If $\mathcal{J}_1$ has enough projectives, then $\mathcal{C}_{12}$ has enough projectives — given by $\mathrm{proj}\,\mathcal{C}_{12} = f_2(\mathrm{proj}\,\mathcal{J}_1)$.

(e) Dually, if $\mathcal{F}_2$ has enogh injectives ...

(f) If $\mathrm{Ext}^1_{\mathcal{A}}(X,Y) \cong \mathrm{Ext}^1_{\mathcal{A}}(Y,X)^* \quad \forall X,Y \in \mathcal{A}$

$\Longrightarrow \quad \mathrm{Proj}\,\mathcal{C}_{12} = \mathrm{inj}\,\mathcal{C}_{12}$

Def: An exact cat. $\mathcal{E}$ is called **Frobenius** if there are enough projectives & injectives and $\operatorname{proj}\mathcal{E} = \operatorname{inj}\mathcal{E}$

Ex: $\mathcal{E} = \operatorname{mod-}\Lambda$, Λ selfinjective, e.g. $\Lambda = \Pi$.

Cor: $\mathcal{A} = \operatorname{mod-}\Pi(\text{Dynkin}) \Rightarrow \mathcal{C}_{12}$ Frobenius

Idea of Pf: $\operatorname{Ext}^1_\Pi(X,Y) \cong \operatorname{Ext}^1_\Pi(Y,X)^*$ (+)

for all $X, Y \in \mathcal{A}$.

If $\mathcal{B} \subseteq \mathcal{A}$ extension-closed subcat., then

$\operatorname{Ext}^1_{\mathcal{B}}(B_1, B_2) = \operatorname{Ext}^1_\Pi(B_1, B_2)$ $\forall B_1, B_2 \in \mathcal{B}$

(+) $\Rightarrow \operatorname{proj}\mathcal{B} = \operatorname{inj}\mathcal{B}$ (∗)

In particular, $(*)$ holds for

$\mathcal{T}_1$, $\mathcal{F}_1$, $\mathcal{C}_{12}$ where $(\mathcal{T}_1, \mathcal{F}_1)$ torsion pair

It follows from [GLS], [Mizuno] that every torsion pair $(\mathcal{C}_w, \mathcal{C}^w) \subseteq \text{mod-}\Pi$ satisfies

$$\mathcal{C}_w = \mathrm{Quot}(t_w(\Pi))$$

$$\mathcal{C}^w = \mathrm{Sub}(f_w(\Pi))$$

Where $f_w(\Pi) \in \mathrm{proj}\, \mathcal{C}^w \overset{(*)}{=} \underline{\underline{\mathrm{inj}\, \mathcal{C}^w}}$

$\Longrightarrow \mathcal{C}^w$ has enough injectives.

Dually, $\mathcal{C}_w$ has enough projectives

$\overset{\text{Prop (d), (e)}}{\Longrightarrow}$ $\mathcal{C}_{v,w} = \mathcal{C}^v \cap \mathcal{C}_w$ is Frobenius

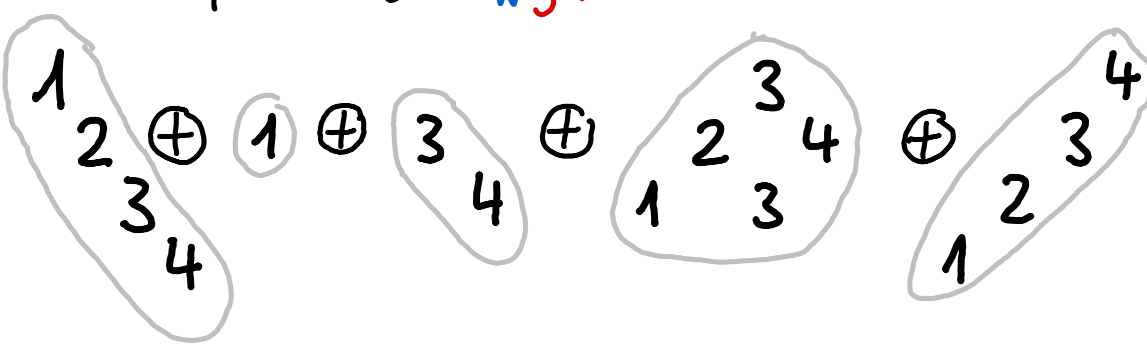
With projective-injective objects

proj $\mathcal{C}_{v,w}$ = add $t_w f_v(\Pi)$ = add $f_v t_w(\Pi)$

<u>Ex:</u> $Q = 1 - 2 - 3 - 4$

$v = (s_2 s_3)^{-1}$, $w = w_0 s_2$

Computing $t_w f_v(\Pi) \cong$



has $5 > 4$ indecomposable direct summands

3. Morita Theorem

for

Frobenius categories

Def: A two-sided Noetherian ring R is **Gorenstein** if $\operatorname{injdim}_R R < \infty$ and $\operatorname{injdim} R_R < \infty$.

Rem: R Gorenstein $\Rightarrow$ $\operatorname{injdim}_R R = \operatorname{injdim} R_R$

Call this **virtual dimension** $\operatorname{virdim} R$.

Ex: i) A selfinjective $\Rightarrow$ $\operatorname{virdim} A = 0$

ii) $\operatorname{gldim} B < \infty$ $\Rightarrow$ $\operatorname{virdim} B = \operatorname{gldim} B$

iii) G gentle algebra $\Rightarrow$ G Gorenstein

Def: For R Gorenstein set

$$GP(R) := \{ M \in \text{mod-}R \mid \mathrm{Ext}_R^{>0}(M,R) = 0 \}$$

category of Gorenstein-projective R-modules.

Ex: i) A selfinjective $\Rightarrow GP(A) = \text{mod-}A$

ii) $\mathrm{gldim}\, B < \infty \Rightarrow GP(B) = \mathrm{proj}\, B$

Prop [cf. Buchweitz] R Gor. $\Rightarrow GP(R)$ Frobenius with $\mathrm{proj}\, GP(R) = \mathrm{proj}\, R$.

Pf uses duality $\mathrm{Hom}_R(-,R) : GP(R) \to GP(R^{op})$ and long exact Ext-sequences

Q: Which Frobenius cats are equivalent to $GP(R)$?

A: **Theorem** (Iyama-K-Wemyss-Yang) (special case)

$\mathcal{E}$ idempotent complete Frobenius cat. s.th.

(i) $\operatorname{proj}\mathcal{E} = \operatorname{add} P$ and $\operatorname{End}_{\mathcal{E}}(P)$ is Noetherian

(ii) $\operatorname{mod}\mathcal{E} = \{F : \mathcal{E} \to Ab \mid$ additive, finitely presented contravariant functors $\}$

and $\operatorname{mod}\mathcal{E}^{op}$ are abelian categories.

The following are equivalent:

(a) $\operatorname{gldim}\operatorname{mod}\mathcal{E} < \infty$ and $\operatorname{gldim}\operatorname{mod}\mathcal{E}^{op} < \infty$.

(b) $\operatorname{Hom}_{\mathcal{E}}(P,-) : \mathcal{E} \xrightarrow{\cong} GP(\operatorname{End}_{\mathcal{E}}(P))$

exact equivalence & $\operatorname{virdim}\operatorname{End}_{\mathcal{E}}(P) \leq \operatorname{gldim}\operatorname{mod}\mathcal{E}$

<u>Ex</u>: Let $\mathcal{E} = \operatorname{proj} S$ where $\operatorname{gldim} S = \infty$.

Then $\mathcal{E} \not\cong GP(R)$ R Gorenstein

since $\operatorname{mod} \mathcal{E} \cong \operatorname{mod-}S$ has

infinite global dimension.

4. Applying the Morita Theorem to Leclerc's Frobenius categories

Lem: $\mathcal{E}$ additive category with kernels and cokernels. Then gldim mod $\mathcal{E}$, gldim mod $\mathcal{E}^{op} \leq 2$

Pf: $F \in \operatorname{mod} \mathcal{E}$

$\overset{\text{def}}{\Longrightarrow} \exists\, (-,X) \xrightarrow{(-,f)} (-,Y) \longrightarrow F \longrightarrow 0$ ex.

$\Longrightarrow 0 \to (-,\ker f) \longrightarrow (-,X) \longrightarrow (-,Y) \to F \to 0$

shows $\operatorname{prdim} F \leq 2$.

As before $(\mathcal{C}_w, \mathcal{C}^w) \subset \operatorname{mod}\Pi(\text{Dynkin})$
torsion pair

<u>Cor</u>: There are Gorenstein rings $\Pi_w, \Pi^w, \Pi_{v,w}$ of virtual dimension ≤ 2 and exact equivalences

(a) $\mathcal{C}_w \cong GP(\Pi_w) \cong \operatorname{Sub}(\Pi_w)$

(b) $\mathcal{C}^w \cong GP(\Pi^w) \cong \operatorname{Sub}(\Pi^w)$

(c) $\mathcal{C}_{v,w} \cong GP(\Pi_{v,w}) \cong \Omega^2(\operatorname{mod-}\Pi_{v,w})$

(a) & (b) identify the GLS-categories (LHS) with the BIRSc-cats (RHS), cf. [GLS].

More precisely, for each vertex $i \in Q_0$ let $I_i := \Pi(1-e_i)\Pi$ two-sided ideal

For $w = \underbrace{s_{i_1} \cdot \ldots \cdot s_{i_k}}_{\text{red. expr.}} \in W$ let

$$I_w = I_{i_1} \cdot \ldots \cdot I_{i_k} .$$

Then $\Pi_w \cong \left(\Pi / I_{w^{-1}}\right)^{op}$ and

$$\Pi^w \cong \left(\Pi / I_{w^{-1} w_0}\right)$$

longest Weyl group elt.

The latter algebras have virdim ≤ 1 [BIRSc]

Hence $GP(-) \cong \Omega(\mathrm{mod}\text{-}) \cong \mathrm{Sub}(-)$

"Condition (P)":

$\mathcal{C}_v \subset \mathcal{C}_w \iff w = v'v$ and $\ell(w) = \ell(v') + \ell(v)$

Proposition (Baumann, Kamnitzer & Tingley)

$(P) \implies \mathcal{C}_{v'} \cong \mathcal{C}_{v,w}$

Cor:

$(P) \implies \Pi_{v,w} \overset{\text{Morita}}{\sim} \Pi_{v'}$

In particular, $\operatorname{virdim} \Pi_{v,w} \leq 1$

5. Concluding remarks and open questions

Open problem:

Give "combinatorial description" of $\Pi_{v,w}$ (e.g. as $\overset{Q_{v,w}}{\text{quiver}}$ with relations)

Rem: $|(Q_{v,w})_0| \gg |Q_0|$ in general

Question (Leclerc)

How does virdim $\Pi_{v,w}$ depend on Q, v, w?

How is virdim $\Pi_{v,w}$ related to geometry of $\mathring{\mathcal{R}}_{v,w}$?

Rem:

See "Partial Answer 12" in arXiv 1709.04785 for partial answers...