

Update on singularity categories

based on arXiv: 2108.03292
(cf. also 2103.06584)

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Motivation
(Representation
theory)

R (noetherian) ring.

Goal: Understand/Classify
all R -modules.

Well-known problem:

This is almost always "hopeless"!

\rightsquigarrow Today, restrict to "understanding"
a subcategory of R -mod:

$$\Omega^\infty(R) := \left\{ M \in R\text{-mod} \mid \begin{array}{l} \text{for all } n \geq 0, \text{ there is} \\ N \in R\text{-mod, s.t.} \\ M \in \text{add-}\Omega^n(N) \end{array} \right\}$$

Think of $\Omega^\infty(R)$ as
 "stable range" / "stabilization"
 of syzygy functor Ω .

Examples:

(a) $\text{gldim } R < \infty \Rightarrow \Omega^\infty(R) = 0.$

(b) $\Omega^\infty\left(\frac{\mathbb{C}[x, y, z]}{(x, y, z)^2}\right) \cong \underline{\text{add}}\text{-}S, \quad S \cong \frac{\mathbb{C}[x, y, z]}{(x, y, z)}$
 $\mathbb{C}(\overset{\mathbb{Q}}{\underset{\mathbb{Q}}{\mathbb{C}}}) / (\text{arrow ideal})^2$ ("wild") simple

(b') $\Omega^\infty\left(\mathbb{C}(\overset{\mathbb{Q}}{\underset{\mathbb{Q}}{\mathbb{C}}} \leftarrow 2) / (\text{arrow ideal})^2\right) \cong \underline{\text{add}}\text{-}S_1$

(c) R selfinjective $\Rightarrow \Omega^\infty(R) \cong \underline{R}\text{-mod.}$

(d) R Gorenstein, i.e. $\text{inj. dim}_R R, R_R < \infty$

$\Rightarrow \Omega^\infty(R) \cong \underline{GP}(R) := \{M \in R\text{-mod} \mid \text{Ext}_R^{>0}(M, R) = 0\}$

In particular, in (c) & (d) $\Omega^\infty(R)$ admits triang. struct.

A finite
representation type

classification for

$\Omega^\infty(R)$, where

R complete local

Gorenstein \mathbb{C} -algebra.

Thm [Auslander-Reiten, Eisenbud, Knörrer, Herzog
Buchweitz-Greuel-Schreyer, ...]

$$R = \mathbb{C}[z_0, \dots, z_d] / I \quad \underline{\text{Gorenstein.}}$$

If $\Omega^\infty(R) \neq 0$ has finite repr. type

THEN $R \cong \mathbb{C}[z_0, \dots, z_d] / (f) =: \mathbb{P}_d / (f)$

is an ADE-hypersurface.

[e.g. $f = z_0^{n+1} + z_1^2 + \dots + z_d^2 \quad (A_n)$]

Moreover, in this case, TFAE:

(1) $\Omega^\infty(\mathbb{P}_d / (f)) \cong \Omega^\infty(\mathbb{P}_e / (g))$ as add. cats

(2) $\Omega^\infty(\mathbb{P}_d / (f)) \cong \Omega^\infty(\mathbb{P}_e / (g))$ as Δ -cats

(3) $|d-e| = 2n$ and $f-g = z_1^2 + \dots + z_{2n}^2$
(after suitable "coordinate change")

Rem:

Today we will generalize
the last part of Thm. to all
isolated hypersurface
singularities!

Matrix

factorizations

Let $0 \neq f \in \mathbb{C}[[z_0, \dots, z_d]] =: S$

A matrix factorization (MF)

of f is a pair

$$(A, B) \in \text{Mat}_n(S) \times \text{Mat}_n(S)$$

satisfying

$$A \cdot B = f \cdot \text{Id}_n = B \cdot A$$

Examples:

$(\text{Id}_n, f \cdot \text{Id}_n)$ and $(f \cdot \text{Id}_n, \text{Id}_n)$

are trivial matrix factorizations

Example: $f = z_0^2 + z_1^2 + z_2^2 + z_3^2$

Step 1: Consider $g = z_0^2 + z_1^2$

$$g = (z_0 + iz_1)(z_0 - iz_1) =: XY \quad \text{is MF}$$

Step 2:

$$\begin{pmatrix} -X & (z_2 - iz_3) \\ (z_2 + iz_3) & y \end{pmatrix}, \begin{pmatrix} -y & (z_2 - iz_3) \\ (z_2 + iz_3) & X \end{pmatrix}$$

is MF for f .

Remark:

(a) A similar approach yields a Matrix fact.

$$A \cdot A = (z_0^2 + z_1^2 + z_2^2 + z_3^2) \cdot \text{Id}_4$$

which is used in Dirac's discovery of the Dirac equation.

(b) More recently, theoretical physicists are interested in understanding and classifying MFs in relation with 2D-Quantum Field Theories.

Example: $f = z_0^2 + z_1^2 + z_2^2 + z_3^2$

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Step 2:

$$\begin{pmatrix} -X & (z_2 - iz_3) \\ (z_2 + iz_3) & Y \end{pmatrix}, \begin{pmatrix} -Y & (z_2 - iz_3) \\ (z_2 + iz_3) & X \end{pmatrix}$$

is MF for f .

Observation (Knörrer 1987):

Let $0 \neq g \in \mathbb{C}[z_0, \dots, z_d]$

The construction in Step 2
defines a map

$$MF(g) \longrightarrow MF\left(g + z_{d+1}^2 + z_{d+2}^2\right)$$

This is a bijection
(up to sums of trivial MFs).

More precisely, MFs (A, B) of f yield 2-periodic diagrams

$$\dots \xrightarrow{A} S^n \xrightarrow{B} S^n \xrightarrow{A} S^n \xrightarrow{B} \dots$$

which (analogous to chain complexes)

form a differential \mathbb{Z} -graded category

$MF(f)$

with degree d maps $m \in \text{Hom}_{MF(f)}^d((A, B), (A', B'))$

(e.g. for $d = -2$: $m = (m_i)_i$)

$$\begin{array}{ccccccccc}
 \dots & \xrightarrow{A} & S^n & \xrightarrow{B} & S^n & \xrightarrow{A} & S^n & \xrightarrow{B} & S^n & \xrightarrow{A} & S^n & \xrightarrow{\dots} \\
 & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & & \\
 & & m_i & & m_{i+1} & & m_{i+2} & & m_{i+3} & & m_{i+4} & \\
 & & & & & & & & & & & \\
 \dots & \xrightarrow{A'} & S^m & \xrightarrow{B'} & S^m & \xrightarrow{A'} & S^m & \xrightarrow{B'} & S^m & \xrightarrow{A'} & S^m & \xrightarrow{\dots}
 \end{array}$$

and differential

$$d(m) = m d_{(A,B)} - (-1)^{|m|} d_{(A',B')} m$$

(where $d_{(A,B)}^i = \begin{cases} A & i \text{ even} \\ B & i \text{ odd} \end{cases}$, similar for $d_{(A',B')}$)

In analogy with homotopy categories of complexes one can define the homotopy category of MFs $[MF(f)]$ with morphism spaces

$$H^0(\text{Hom}_{MF(f)}^\bullet((A,B), (A',B'))).$$

Thm (Knörrer '87) There is a triangle equiv.

$$[MF(f)] \xrightarrow{\sim} [MF(f + z_{d+1}^2 + z_{d+2}^2)]$$

The
Buchweitz – Orlov
Singularity category

X quasi-proj. variety / \mathbb{C}

$$\text{Perf}(X) \hookrightarrow D^b(\text{Coh } X) \twoheadrightarrow D_{\text{sg}}(X) := \frac{D^b(\text{Coh } X)}{\text{Perf}(X)}$$

'smooth part'
consisting of
bounded complexes of
vector bundles

Singularity Category
measures
complexity of
singularities of X

Thm (Auslander - Buchsbaum & Serre)

$$X \text{ smooth} \iff D_{\text{sg}}(X) = 0$$

Thm (Orlov) If X has isolated singularities

$$D_{\text{Sg}}(X) \cong \bigoplus_{s \in \text{Sing}(X)} D_{\text{Sg}}(\hat{\mathcal{O}}_s)$$

(up to taking direct summands)

where

$$D_{\text{Sg}}(\mathbb{R}) := \frac{D^b(\text{mod-}\mathbb{R})}{K^b(\text{proj-}\mathbb{R})}$$

singularity category of a

noetherian ring \mathbb{R} .

Thm (Buchweitz, Eisenbud)

Let $0 \neq f \in \mathbb{C}[z_0, \dots, z_d] =: S$

$$\Omega^\infty(S/(f)) \cong [MF(f)] \cong D_{sg}(S/(f))$$

Def: Two (possibly noncomm.) algebras R, S are called singular equivalent

if there is a triangle equivalence

$$D_{sg}(R) \cong D_{sg}(S)$$

Complete list of known
singular equivalences between
commutative

Complete local \mathbb{C} -algebras:

$$(0) D_{\text{sg}}(R) = 0 = D_{\text{sg}}(S), \text{ if } \text{gldim } R, S < \infty$$

$$(1) D_{\text{sg}}\left(\frac{P_d}{(S)}\right) \cong D_{\text{sg}}\left(\frac{P_d[y_1, \dots, y_{2n}]}{(f + y_1^2 + \dots + y_{2n}^2)}\right), \text{ for } f \neq 0$$

[Knörrer '87]

Let $\mathbb{C}[z_1, \dots, z_d]_{\frac{1}{m}(a_1, \dots, a_d)}$ be the invariant ring of the following group action:

$$z_i \longmapsto \varepsilon_m^{a_i} z_i \quad \left(\begin{array}{l} \text{for a primitive} \\ m\text{-th root of unity} \\ \varepsilon_m \in \mathbb{C} \end{array} \right)$$

$$(2) D_{Sg} \left(\mathbb{C}[z_1, z_2]^{\frac{1}{m}(1,1)} \right) \simeq D_{Sg} \left(\frac{\mathbb{C}[z_1, \dots, z_{m-1}]}{(z_1, \dots, z_{m-1})^2} \right)$$

[D. Yang, Y. Kawamata, K. Karmazyn]
all ~ 2015

$$(3) D_{Sg} \left(\mathbb{C}[z_1, z_2, z_3]^{\frac{1}{2}(1,1,1)} \right) \simeq D_{Sg} \left(\mathbb{C}[z_1, z_2]^{\frac{1}{4}(1,1)} \right)$$

[K. 2021]

Observation:

Knörrer's equivalences are
the only known non-trivial
Gorenstein examples.

Goal:

Explain this observation
(to some extent), i.e.

explain why we (maybe)
shouldn't expect to find (many)
more equivalences.

1. Step:

Reduction to

hypersurfaces

Thm [cf. K.'21 and also Puthenpurakal '21]

$R_j = P_{d_j}/I_j$ complete local \mathbb{C} -algebras, s.th

$$D_{\text{sg}}(R_1) \cong D_{\text{sg}}(R_2) \text{ as } \Delta\text{-ted cats}$$

and R_1 is Gorenstein with isolated sing.

If $\left\{ \begin{array}{l} R_1 = P_{d_1}/(f_1) \text{ hypersurface } \underline{\text{OR}} \\ \text{krdim } R_1 \neq \text{krdim } R_2 \end{array} \right.$

Then both $R_j \cong P_{d_j}/(f_j)$ are hypersurfaces
(with isolated singularities)

2. Step :

Classifying dg singularity
categories of hypersurfaces
up to quasi-equivalence

DG-singularity categories

For hypersurface singularities g
have seen dg cat. $MF(g)$
s.th. $[MF(g)] \cong D_{sg}(\mathbb{P}^d/(g))$.

More generally, can define
 $D_{sg}^{dg}(R)$ as dg quotient [Keller, Drinfeld]
 $D_{dg}^b(R\text{-mod}) / \text{Perf}_{dg}(R)$

and show

$$MF(g) \underset{\text{quasi-eg.}}{\cong} D_{sg}^{dg}(\mathbb{P}^d/(g))$$

[Blanc-Robalo-Toën-Verzozzi]

The following
result answers
a question of
Keller & Skinder.

Thm [K. '21]

Let $0 \neq f_j \in \mathbb{P}_{d_j}$, $d_1 \geq d_2$ and assume that f_1 has isolated singularity.

TFAE

(a) $D_{sg}^{dg}(\mathbb{P}_{d_1}/(f_1)) \cong D_{sg}^{dg}(\mathbb{P}_{d_2}/(f_2))$ \mathbb{C} -linear quasi-equiv. of dg-cat's.

(b) $\mathbb{P}_{d_1}/(f_1) \cong \mathbb{P}_{d_1}/(g_1)$, s. th.

$$g_1 - f_2 = z_{d_2+1}^2 + \dots + z_{d_1}^2 \quad \underline{\text{and}} \quad d_1 - d_2 = 2n.$$

Pf (Sketch) (b) \Rightarrow (a) [Knörrer; ^{dg version:} Dyckerhoff and also Orlov]

$T(f) := \mathbb{P}_d/(f, \partial_0 f, \dots, \partial_d f)$ Tyurina algebra

(a) $\xRightarrow{\text{[Hua-Keller]}}$ $T(f_1) \cong T(f_2)$

Using Hochschild cohomology of dg categories, which is equiv. to Buchweitz's singular Hochschild cohom.

by work of Keller inspired by Zhengfang Wang's thesis. \perp

Fact: $T(f) \cong T(f + y_1^2 + \dots + y_t^2)$

In combination with discussion above, we get

$$T(f_1) \cong T(f_2) \cong T(\underbrace{f_2 + z_{d_2+1}^2 + \dots + z_{d_1}^2}_{=: g_1 \in \mathbb{P}_{d_1}})$$

Thm (Mather-Yau, cf. also Greuel-Pham)

$h_i \in \mathbb{P}_d$, s.t. $T(h_1) \cong T(h_2)$ as algebras
(isol. sing.)

$$\Rightarrow \mathbb{P}_d / (h_1) \cong \mathbb{P}_d / (h_2)$$

Applying this to $f_1, g_1 \in \mathbb{P}_{d_1}$ yields

$$\mathbb{P}_{d_1} / (f_1) \cong \mathbb{P}_{d_1} / (f_2 + z_{d_2+1}^2 + \dots + z_{d_1}^2)$$

Using Knörrer's work one can check that

$$D_{\text{Sg}}(P_d/(h)) \not\cong D_{\text{Sg}}(P_{d+1}/(h+z_{d+1}^2))$$

implying $d_1 - d_2$ is even and

Completing the proof.

Rem:

(a) Let $p \in \mathbb{Z}$ be prime. There are Δ -equiv.

$$D_{\text{Sg}}(\mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}\text{-mod} \cong \mathbb{Z}/p\text{-mod} \cong \frac{\mathbb{Z}/p[x]}{(x^2)}\text{-mod} \cong_{\text{Sg}} \left(\frac{\mathbb{Z}/p[x]}{(x^2)} \right)$$

However, there are dg-enhancements

which are not quasi-equivalent if $p \neq 2$

(Schlichting '02)

(b) This does not happen for connected representation finite categories over alg. closed fields! [Muro '18, cf. also Keller '18 and K.-Yang '16]

In particular, if $D_{\text{sg}}(R_i)$ is connected and representation-finite for k -algebras R_i with $k = \bar{k}$. Then TFAE

$$(1) \quad D_{\text{sg}}(R_1) \cong D_{\text{sg}}(R_2) \quad \text{as } \Delta\text{-ted Categories}$$

$$(2) \quad D_{\text{sg}}^{\text{dg}}(R_1) \cong D_{\text{sg}}^{\text{dg}}(R_2) \quad \text{quasi-equiv. as dg-cat}$$

Using this together with our Theorem above, can recover parts of finite type classification result in the beginning.