Knörrer invariant algebras & the Euclidean algorithm

(based on jt. work with Joe Karmazyn)

Martin Kalck

University of Edinburgh

ARTIN 47, Glasgow

April 8, 2016

G : $\subset GL(2, \mathbb{C})$

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(Non-)commutative resolutions of singularities

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(relative) singularity categories

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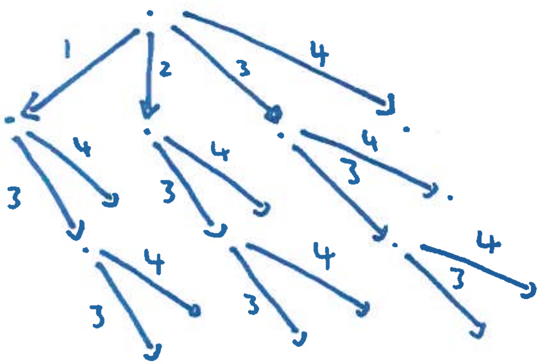

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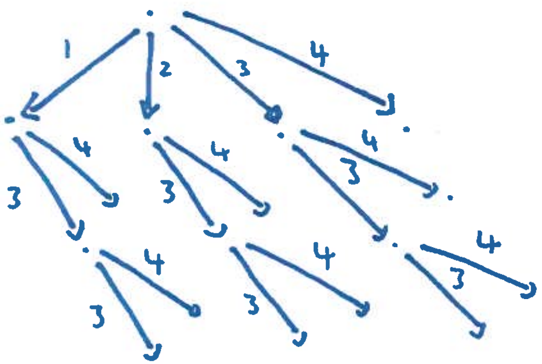

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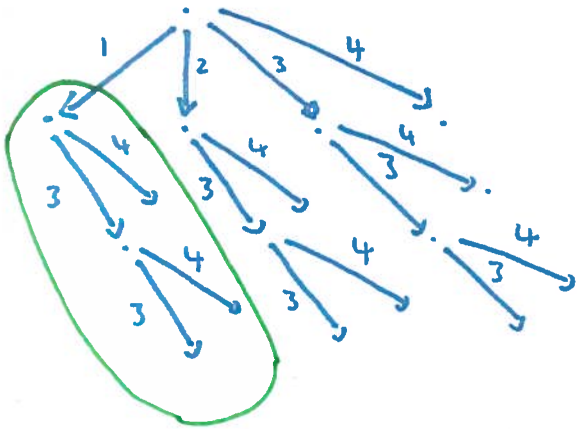

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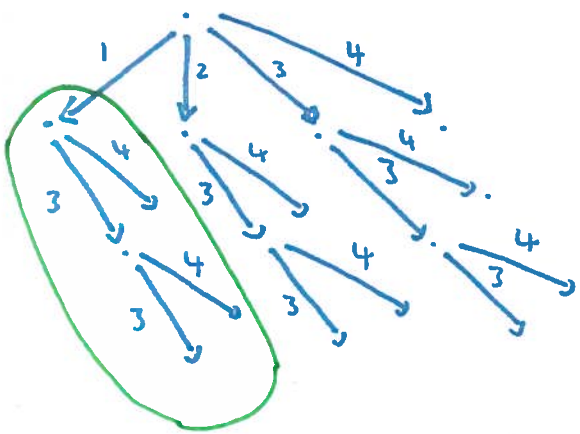

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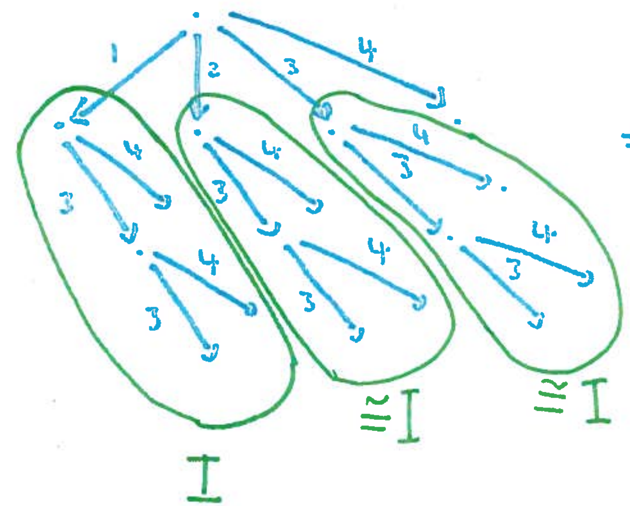

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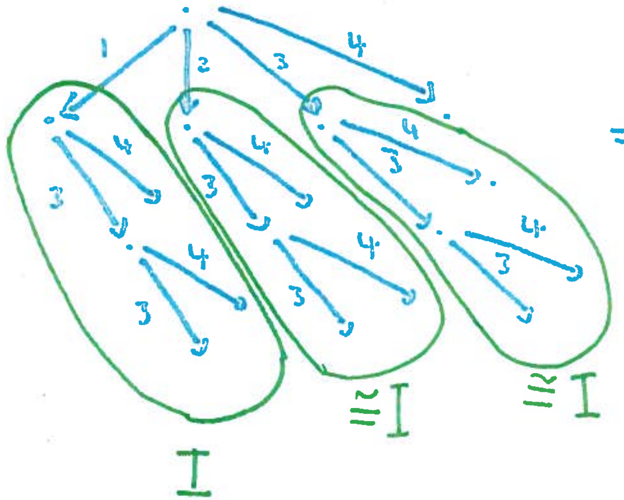

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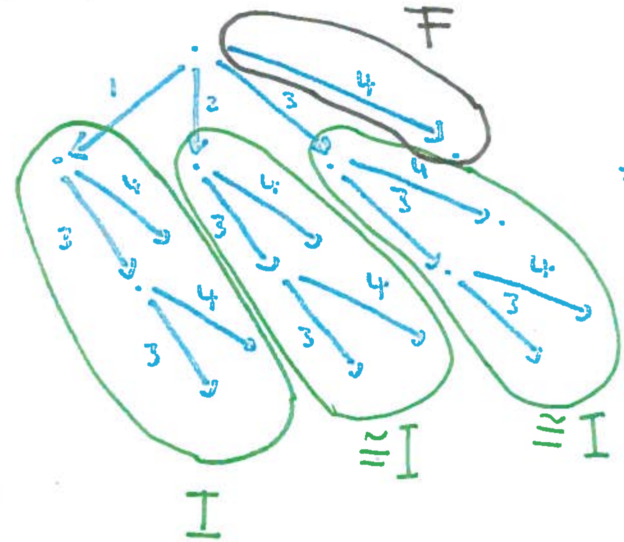

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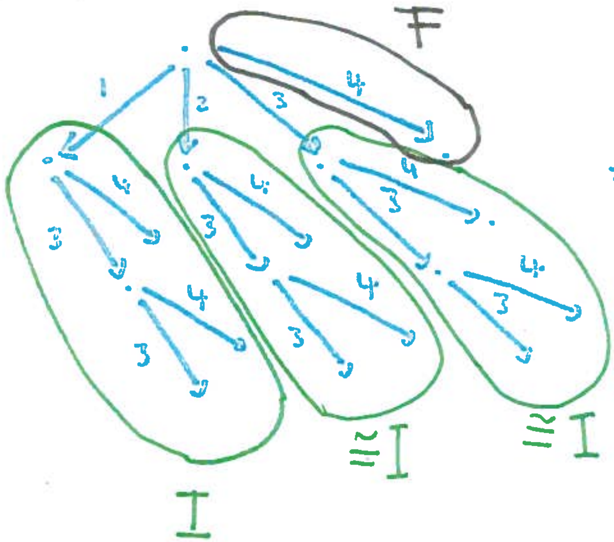

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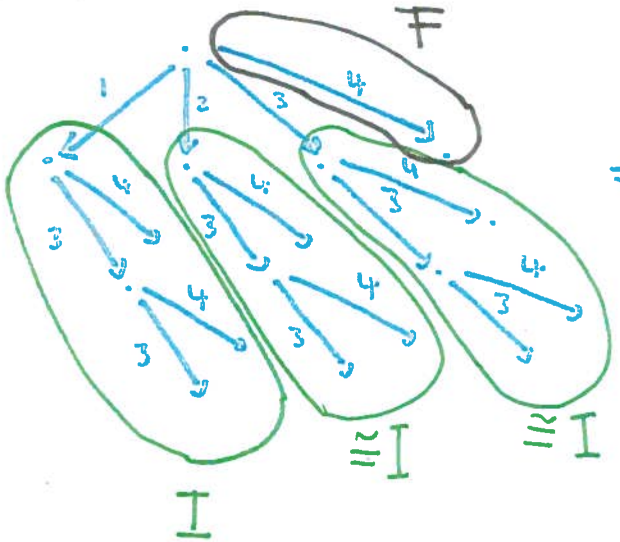

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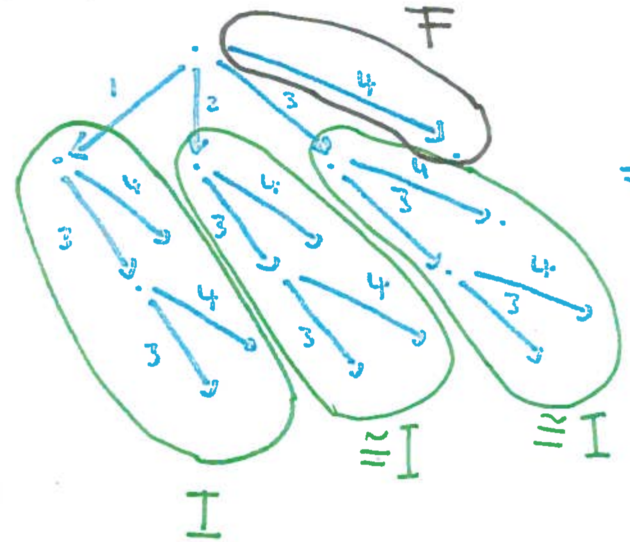

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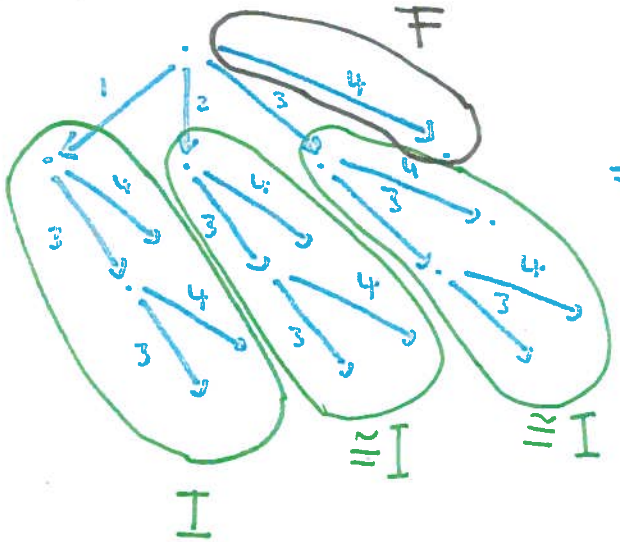

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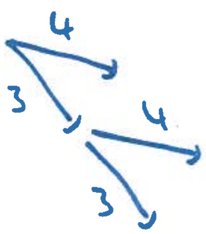

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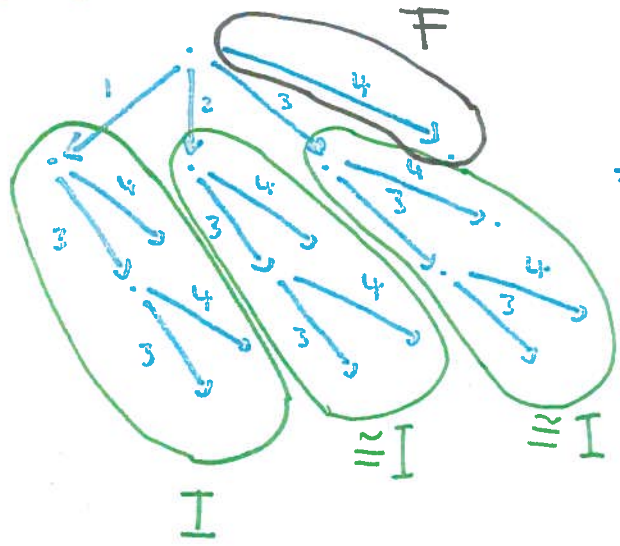

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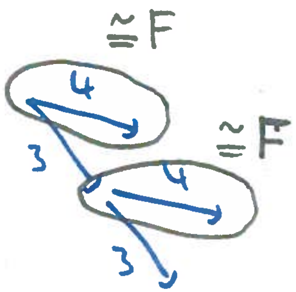

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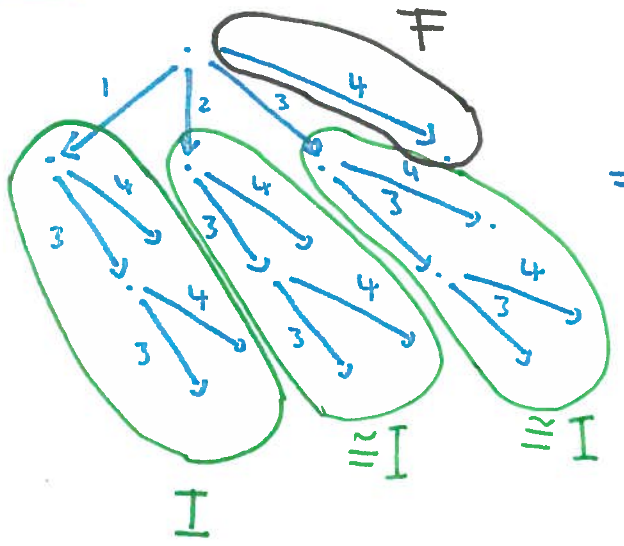

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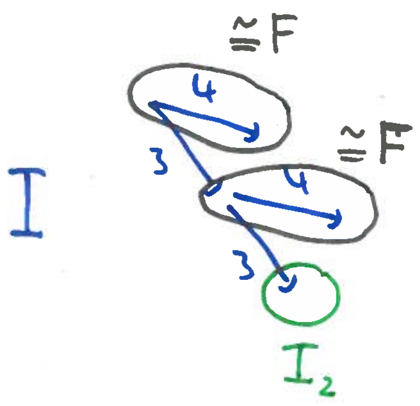

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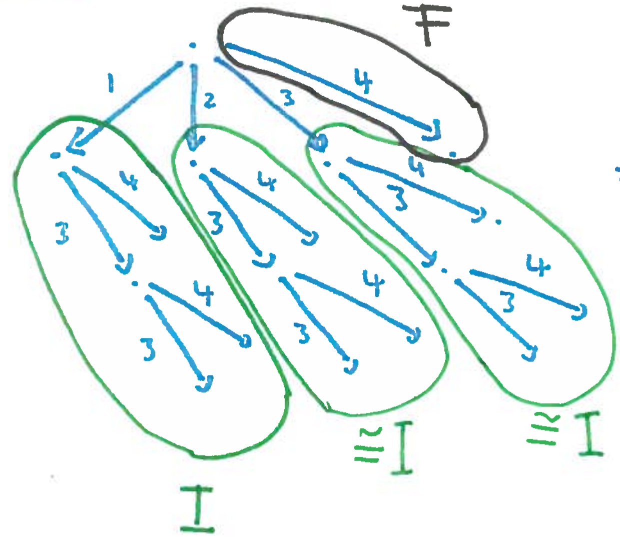

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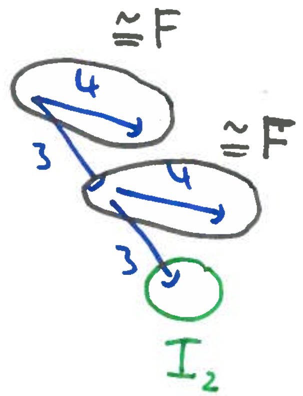

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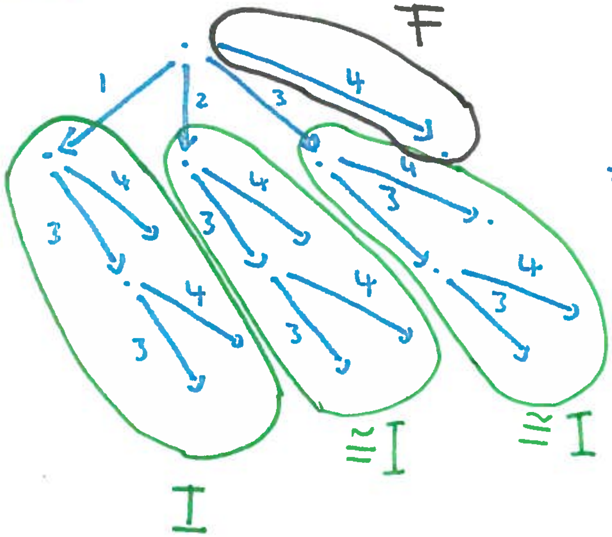

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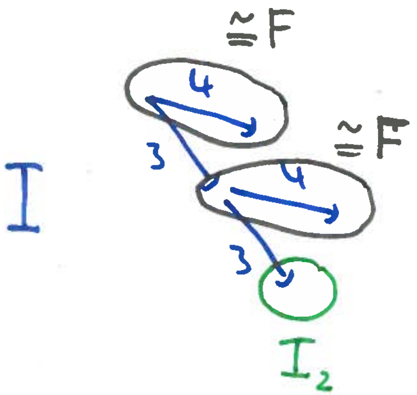

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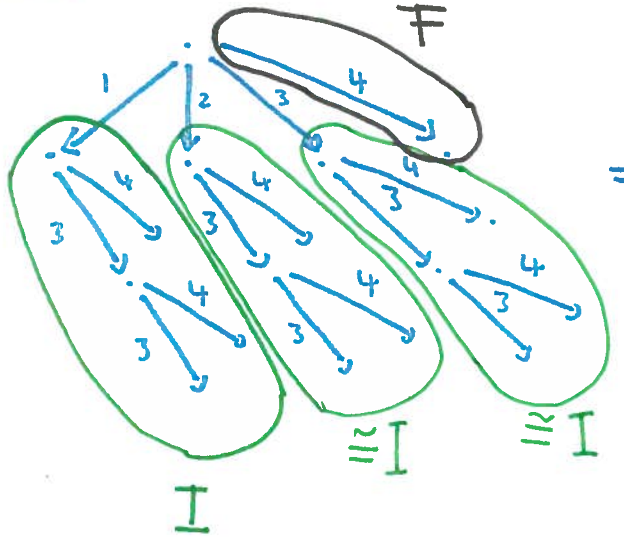

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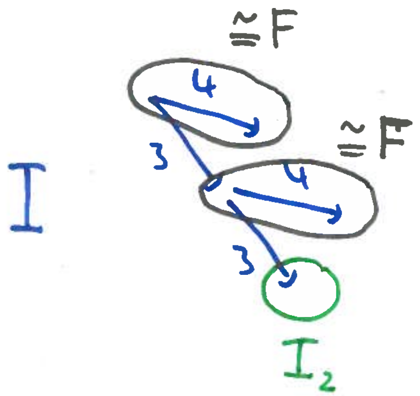
<u>Euclidean algorithm</u>

$$\left.\begin{aligned} &\dim K_{r,a} = r = 17 \\ &\dim I = a = 5 \\ &[K_{r,a} : I] = 3 \\ &\dim F = 2 \end{aligned}\right\}$$

$$\dim K_{r,a} = [K_{r,a} : I] \cdot \dim I + \dim F$$

$$17 = 3 \cdot 5 + 2$$

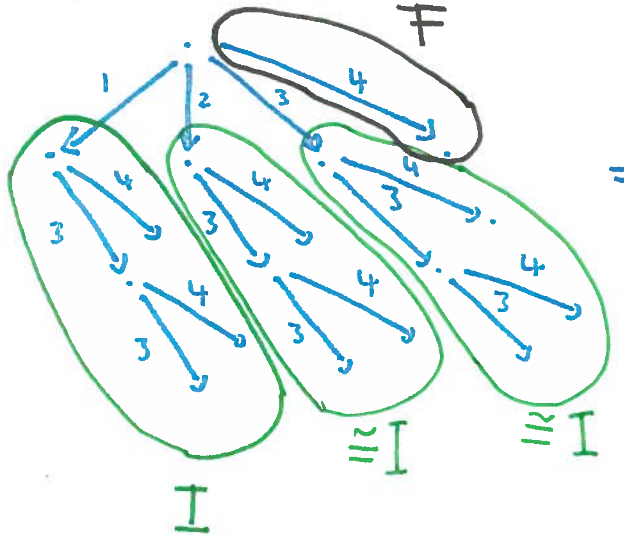


$$\left.\begin{aligned} &[I : F] = 2 \\ &\dim I_2 = 1 \end{aligned}\right\}$$

$$\dim I = [I : F] \cdot \dim F + \dim I_2$$

More generally, $\mathbb{C}$-dimensions & multiplicities of subfactors of $K_{r,a}$ encode <u>Euclidean algorithm</u> for (r,a).

<u>Ex</u>: $r=17$, $a=5$



$=: K_{r,a}$

$\dim K_{r,a} = r = 17$

$\dim I = a = 5$

$[K_{r,a} : I] = 3$

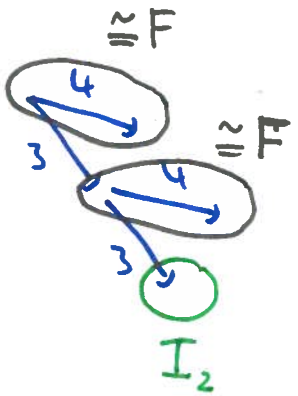
$\dim F = 2$

<u>Euclidean algorithm</u>

$\dim K_{r,a} = [K_{r,a} : I] \cdot \dim I + \dim F$

$17 = 3 \cdot 5 + 2$

I



$[I : F] = 2$

$\dim I_2 = 1$

$\dim I = [I : F] \cdot \dim F + \dim I_2$

$5 = 2 \cdot 2 + 1$