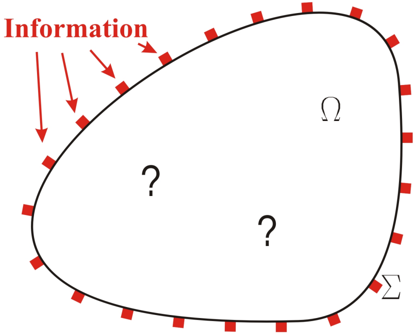
Shape and Topology Optimization Techniques in EIT

Martin Kanitsar

28. August 2009

Detect objects in a domain Ω with boundary Σ



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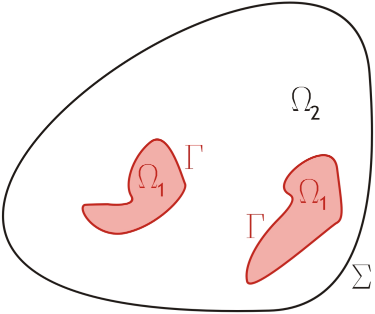

Concept of Electrical Impedance Tomography

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Concept of Electrical Impedance Tomography

- background and objects have different physical properties, i.e. conductivity values
- apply current through electrodes on Σ
- take measurements of the electrical potential on Σ
- use an appropriate model to describes the physical situation

Applications

- medicine
  - blood clots in the lungs
  - breast cancer
- geophysics
  - pollutant plumes
  - oil inclusions

Mathematical model

$$
\begin{aligned}
-\operatorname{div}(q\nabla u) &= 0 \text{ in } \Omega, && (1.1)\\
q\partial_n u &= f \text{ on } \Sigma, && (1.2)\\
\int_\Sigma f(x) &= 0. && (1.3)
\end{aligned}
$$

u ... electrical potential
q ... conductivity
f ... applied current

Relationship to Maxwell's equations

Maxwell's equation in a macroscopic time harmonic field:

$$\nabla \times H = J - i\omega D \tag{1.4}$$

H ... magnetic field	D ... displacement	ω ... angular freq.
E ... electrical field	J ... current density	ϵ ... permittivity

Relationship to Maxwell's equations

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- medium reacts linearly : $D = \epsilon E$
- Ohm's law : $J = qE$
- relationship :$E = -\nabla u$

$$0 = \operatorname{div}(\nabla \times H) = -\operatorname{div}((q - i\omega\epsilon)\nabla u)\,. \qquad (1.5)$$

Consider real part of (1.5): $-\operatorname{div}(q\nabla u) = 0\,.$

Mathematical model

The so-called forward problem

$$-\operatorname{div}(q\nabla u) = 0 \text{ in } \Omega, \tag{1.6}$$

$$q\partial_n u = f \text{ on } \Sigma, \tag{1.7}$$

$$\int_\Sigma f(x) = 0, \tag{1.8}$$

u ... electrical potential
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Mathematical model

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\int_\Sigma f(x) &= 0, &(1.8)
\end{aligned}
$$

with a normalization restriction

$$\int_\Sigma u - m = 0. \tag{1.9}$$

u ... electrical potential
q ... conductivity
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Regularized output least squares problem

$\{m_i\}_{i=1}^{M}$...measurements on Σ,
M... number of measurements

We have to minimize

$$J(u,q) = \frac{1}{2}\sum_{i=1}^{M}\int_{\Sigma}|u_i - m_i|^2 ds + \beta\, R(q), \qquad (2.1)$$

$$\text{s.t.}(1.6) - (1.9)$$

with $u = (u_1, u_2, ...u_M)$ and $R(q) = \int_{\Omega}\left(|\nabla q|^2 + \epsilon\right)^{\frac{1}{2}} dx$

Reduction Process

Instead of the constraint minimization problem

$$\min_{(u,q)} J\,(u,q)$$
$$\text{s.t.}(1.6)-(1.9)$$

we consider the unconstraint minimization problem

$$\min_{q} \hat{J}\,(q)$$

with

$$\hat{J}(q) = J(u(q),q)$$

Representation of q

Assume conductivity is piecewise constant:

$$q(x) = \begin{cases} q_1 & \forall\ x \in \Omega_1, \\ q_2 & \forall\ x \in \Omega_2. \end{cases}$$

$$q = q(q_1, q_2, \Omega_1, \Omega_2) \overset{\Omega_1 = \Omega \setminus \Omega_2}{=} q(q_1, q_2, \Omega_2)$$

$$\phi(x) = \begin{cases} > 0 & \forall\ x \in \Omega_1, \\ < 0 & \forall\ x \in \Omega_2. \end{cases}$$

$$q(q_1, q_2, \phi) = q_1 \mathcal{H}^{\epsilon}\left(\phi(x)\right) + q_2 \left(1 - \mathcal{H}^{\epsilon}\left(\phi(x)\right)\right). \tag{2.2}$$

Shape gradient

$$\hat{J}'\,(q)\,[\tilde{q}] = \langle -\nabla p \nabla u - \beta r(q), \tilde{q} \rangle_{\Omega}\,. \tag{2.3}$$

with the solution p of the adjoint problem

$$\begin{aligned} -\mathrm{div}\,(q\nabla p) &= 0 && \text{in } \Omega, \\ q\partial_n p &= u - m && \text{on } \Sigma, \\ \int_{\Omega} p &= 0. \end{aligned}$$

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$$\begin{aligned} q(q_1, q_2, \phi) &= q_1 \mathcal{H}^{\epsilon}(\phi) + q_2 \left(1 - \mathcal{H}^{\epsilon}(\phi)\right). \\ q_{\phi}\left[\tilde{\phi}\right] &= (q_1 - q_2)\delta^{\epsilon}\left(\tilde{\phi}\right), \qquad q_{q_1} = \mathcal{H}^{\epsilon}(\phi). \end{aligned} \tag{2.4}$$

Shape gradient

Use the chain rule:

$$\begin{aligned}\hat{J}_{\phi}(q)\left[\tilde{\phi}\right] &= \hat{J}'(q)\left[q_{\phi}\left(\tilde{\phi}\right)\right] \\ &= \int_{\Omega} -\left(\nabla p \nabla u + \beta r(q)\right)(q_1 - q_2)\,\delta^{\epsilon}\left(\tilde{\phi}\right) \\ &=: \int_{\Omega} \hat{\mathcal{J}}_{\phi}\left(\tilde{\phi}\right).\end{aligned}$$

$$\hat{J}_{q_1}(q) = \int_{\Omega} -\left(\nabla p \nabla u + \beta r(q)\right)\left(\mathcal{H}^{\epsilon}(\phi)\right).$$

Algorithm

1. Choose ϕ^0, q_1^0, q_2^0 ; $k_{\max} \in \mathbb{N}$. Set $k = 0$.
2. Compute q^k, u^k, p^k.
3. Compute $\hat{\mathcal{J}}_\phi(\phi^k, q_1^k, q_2^k)$ and set $\phi^{k+1} = \phi^k - \alpha_\phi^k \hat{\mathcal{J}}_\phi$.

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4. Repeat step 2, but with ϕ^{k+1}.
5. Compute $\hat{J}_{q_i}(\phi^{k+1}, q_1^k, q_2^k)$ and set $q_i^{k+1} = q_i^{k+1} - \alpha_i^k \hat{J}_{q_i}$.

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6. Perform a distance re-initialization of ϕ^{k+1}, set $k = k + 1$.
7. Repeat 2-6 until $k = k_{\max}$.

Definition Topological Expansion

In the following we fix q_1 and q_2

$$q(q_1, q_2, \Omega_2) = q(\Omega_2) \tag{3.1}$$

and use $\beta = 0$. Therefore the cost functional becomes

$$J(u(\Omega_2), q(\Omega_2)) = J(\Omega_2).$$

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$$J\left(u\left(\Omega_2\right), q\left(\Omega_2\right)\right) = J\left(\Omega_2\right).$$

The topological expansion $\mathcal{T}^{\varepsilon}$ of J at Ω is defined as

$$\mathcal{T}^{\varepsilon}(\hat{x}) = \frac{J\left(\Omega_2^{\varepsilon}\right) - J\left(\Omega\right)}{\left|\Omega_1^{\varepsilon}\right|}.$$

with $\Omega_1^{\varepsilon} = B(\hat{x}, \varepsilon)$, $\Omega_2^{\varepsilon} = \Omega \backslash \Omega_1^{\varepsilon}$. The topological derivative $\mathcal{T}$ is defined by $\mathcal{T} = \lim\limits_{\varepsilon \to 0} \mathcal{T}^{\varepsilon}$, if the limit exists.

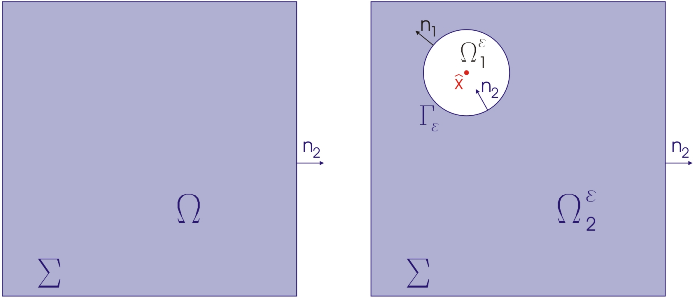

Figure: topological expansion domain

Coupled System

The solution u^ε of

$$\begin{aligned} -\mathrm{div}(q\nabla u^\varepsilon) &= 0 \text{ in } \Omega, \\ q\partial_n u^\varepsilon &= f \text{ on } \Sigma. \end{aligned}$$

can be written as

$$u^\varepsilon = u_1^\varepsilon \mathbb{1}_{\Omega_1^\varepsilon} + u_2^\varepsilon \mathbb{1}_{\Omega_2^\varepsilon},$$

with u_1^ε and u_2^ε the solutions of the coupled system

$$\begin{aligned} -\Delta u_2^\varepsilon &= 0 \text{ in } \Omega_2^\varepsilon, \\ q_2\partial_{n_2} u_2^\varepsilon &= f \text{ on } \Sigma, \\ q_2\partial_{n_1} u_2^\varepsilon &= q_1\partial_{n_1} u_1^\varepsilon \text{ on } \Gamma_\varepsilon, \end{aligned} \qquad \begin{aligned} -\Delta u_1^\varepsilon &= 0 \text{ in } \Omega_1^\varepsilon, \\ u_1^\varepsilon &= u_2^\varepsilon \text{ in } \Gamma_\varepsilon, \end{aligned}$$

Expansion of u_2^ε

$$u_2^\varepsilon = u|_{\Omega_2^\varepsilon} + \mathcal{R}_2^\varepsilon$$

$$\begin{aligned} -\Delta u &= 0 \text{ in } \Omega, \\ q_2 \partial_{n_2} u &= f \text{ on } \Sigma. \end{aligned}$$

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$$\begin{aligned} -\Delta \mathcal{R}_2^\varepsilon &= 0 \text{ in } \Omega_2^\varepsilon, && (3.2) \\ q_2 \partial_{n_2} \mathcal{R}_2^\varepsilon &= 0 \text{ on } \Sigma, && (3.3) \\ q_2 \partial_{n_1} \mathcal{R}_2^\varepsilon &= q_1 \partial_{n_1} u_1^\varepsilon - q_2 \partial_{n_1} u \text{ on } \Gamma_\varepsilon. && (3.4) \end{aligned}$$

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Main Idea to approximate $\mathcal{R}_2^\varepsilon$ by $\tilde{\mathcal{R}}_2^\varepsilon$

- compute solution of (3.2) and (3.4)
- compensate error on Σ by considering (3.2) and (3.3)

Expansion of $J(\Omega_2^\varepsilon)$

$$u_2^\varepsilon(x) = u|_{\Omega_2^\varepsilon} + \mathcal{R}_2^\varepsilon = u|_{\Omega_2^\varepsilon} + \tilde{\mathcal{R}}_2^\varepsilon + \text{Rest} \tag{3.5}$$

$$\begin{aligned}
J(\Omega_2^\varepsilon) &= \frac{1}{2}\int_\Sigma (u_2^\varepsilon - m)^2 \overset{(3.5)}{=} \frac{1}{2}\int_\Sigma \left((u-m) + \left(\tilde{\mathcal{R}}_2^\varepsilon + \text{Rest}\right)\right)^2 \\
&= \frac{1}{2}\int_\Sigma (u-m)^2 \\
&+ \frac{1}{2}\int_\Sigma 2(u-m)\left(\tilde{\mathcal{R}}_2^\varepsilon + \text{Rest}\right) + \left(\tilde{\mathcal{R}}_2^\varepsilon + \text{Rest}\right)^2 \\
&\approx J(\Omega) + |\Omega_1^\varepsilon|\left(\mathcal{T}_0 + \mathcal{T}_1^\varepsilon + \mathcal{T}_2^\varepsilon + \mathcal{T}_3^\varepsilon + \mathcal{T}_4^\varepsilon\right)
\end{aligned}$$

$$\mathcal{T}^{\varepsilon} = \frac{J\left(\Omega_2^{\varepsilon}\right) - J\left(\Omega\right)}{\left|\Omega_1^{\varepsilon}\right|} \approx \mathcal{T}_0 + \mathcal{T}_1^{\varepsilon} + \mathcal{T}_2^{\varepsilon} + \mathcal{T}_3^{\varepsilon} + \mathcal{T}_4^{\varepsilon}$$

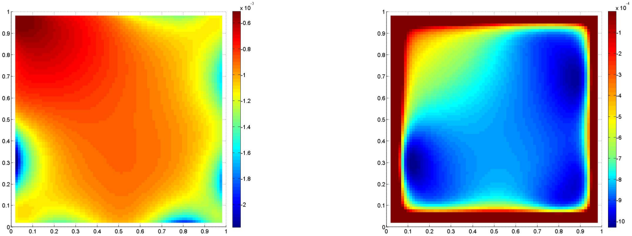

Figure: top view of $\mathcal{T}$ and $\mathcal{T}^{\varepsilon}$

Discretization scheme

Necessary conditions for a distance function

$$
\begin{aligned}
|\nabla\phi(x)| &= 1 \quad x \in \Omega, &(1.1)\\
\phi(x) &= 0 \quad x \in \Gamma &(1.2)
\end{aligned}
$$

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\end{aligned}
$$

Discretized equation of (1.1)

$$
\max\left(\phi_{i,j}-\phi_{i_{\min},j},0\right)^2+\max\left(\phi_{i,j}-\phi_{i,j_{\min}},0\right)^2=h^2 \qquad (1.3)
$$

$$
\phi_{i_{\min},j}=\min\left(\phi_{i-1,j},\phi_{i+1,j}\right) \quad \text{and} \quad \phi_{i,j_{\min}}=\min\left(\phi_{i,j-1},\phi_{i,j+1}\right)
$$

Fast sweeping algorithm

- Initialization

$$\phi_{i,j} = \begin{cases} 0 & \text{if } x_{i,j} \in \Gamma \\ c & \text{otherwise} \end{cases}$$

with c being an upper boundary of the true solution

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with c being an upper boundary of the true solution

- Update
  - Compute solution $\bar{\phi}$

$$\max\left(\bar{\phi} - \phi_{i_{\min},j}, 0\right)^2 + \max\left(\bar{\phi} - \phi_{i,j_{\min}}, 0\right)^2 = h^2$$

  - Set

$$\phi_{i,j} = \min\left(\phi_{i,j}, \bar{\phi}\right).$$

We sweep the whole domain with four alternating orderings:

$$(1)\ \ i = 1 : I,\ j = 1 : J, \qquad (2)\ \ i = I : 1,\ j = 1 : J,$$
$$(3)\ \ i = I : 1,\ j = J : 1, \qquad (4)\ \ i = 1 : I,\ j = J : 1.$$

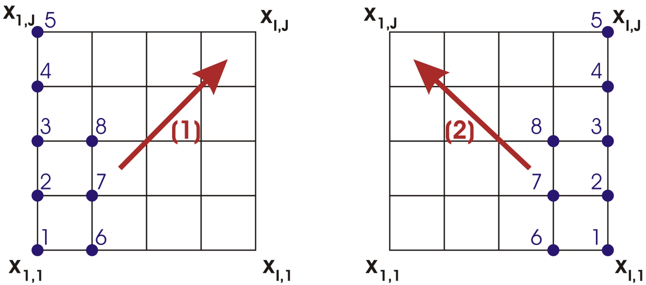


Illustration of 4 sweeps

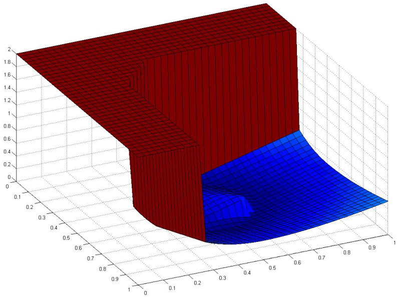


Figure: after 1 sweep

Illustration of 4 sweeps

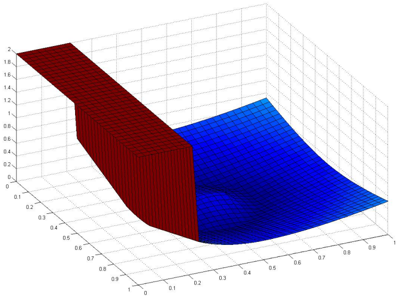


Figure: after 2 sweeps

Illustration of 4 sweeps

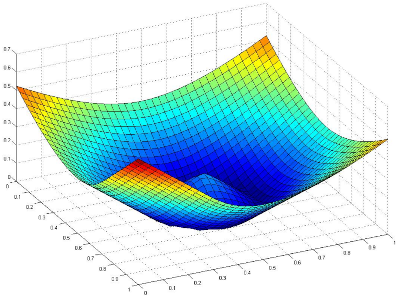


Figure: after 3 sweeps

Illustration of 4 sweeps

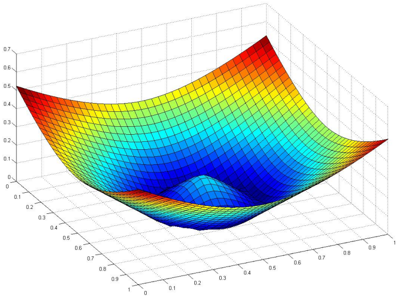


Figure: after 4 sweeps

Basic theoretical results

Theorem

The iterative solution by the fast sweeping algorithm converges monotonically to the solution of the discretized system.

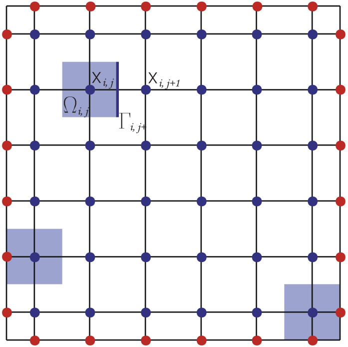
Theorem

Let $\Gamma = \{\boldsymbol{x}_0\}$ and $\phi_{i,j}$ be the solution of the discretized system. The fast sweeping algorithm converges after 4 sweeps to the solution $\phi_{i,j}$ which satisfies

$$d(\boldsymbol{x}_{i,j}) \leq \phi_{i,j} \leq d(\boldsymbol{x}_{i,j}) + O\left(|h \log h|\right).$$

Elliptic equation with Neumann boundary constraints

$$
\begin{aligned}
-\mathrm{div}(q\nabla u) &= 0 \ \text{ in } \Omega, \\
q\partial_n u &= f \ \text{ on } \Sigma, \\
\int_\Sigma f(x) &= 0.
\end{aligned}
$$

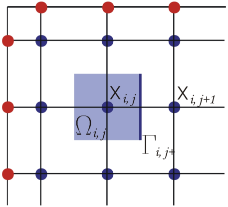


$$\Omega_{i,j} = \left(x_{i-\frac{1}{2},j}, x_{i+\frac{1}{2},j}\right) \times \left(x_{i,j-\frac{1}{2}}, x_{i,j+\frac{1}{2}}\right)$$

Discretization scheme

Gauss' Theorem

$$0 = \int_{\Omega_{i,j}} \operatorname{div}(q\nabla u) = \int_{\Gamma_{i_+,j} \cup \Gamma_{i_-,j} \cup \Gamma_{i,j_+} \cup \Gamma_{i,j_-}} q\partial_n u. \qquad (2.1)$$



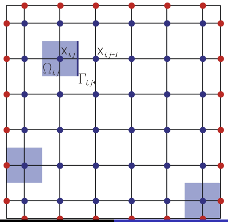
Numerical approximation:

$$\int_{\Gamma_{i,j_+}} q\, \partial_n u \quad \approx \quad (u_{i,j+1} - u_{i,j})q_{i,j_+}$$

Resulting equations

$$u_{i-1,j}\, q_{i_-,j} + u_{i+1,j}\, q_{i_+,j} + u_{i,j-1}\, q_{i,j_-} + u_{i,j+1}\, q_{i,j_+}$$
$$-u_{i,j}\, \left(q_{i_-,j} + q_{i_+,j} + q_{i,j_-} + q_{i,j_+}\right) = 0.$$

$$u_{i,3}\, q_{i,3_-} + u_{i-1,2}\, q_{i_-,2} + u_{i+1,2}\, q_{i_+,2}$$
$$-u_{i,2}\, \left(q_{i,3_-} + q_{i_-,2} + q_{i_+,2}\right) = -\int_{x_i-\frac{h}{2}}^{x_i+\frac{h}{2}} f(x,0)dx$$



Structure of the Matrix

$$L_h \cdot u_h = d_h. \tag{2.2}$$

$$L_h(1:8,1:8) =$$

$$\left(\begin{array}{cccc|cccc}
* & q_{2_+,2} & & & q_{2,2_+} & & & \\
q_{3_-,2} & * & q_{3_+,2} & & & q_{3,2_+} & & \\
 & q_{4_-,2} & * & q_{4_+,2} & & & q_{4,2_+} & \\
 & & q_{5_-,2} & * & & & & q_{5,2_+} \\
\hline
q_{2,3_-} & & & & * & q_{2_+,3} & & \\
 & q_{3,3_-} & & & q_{3_-,3} & * & q_{3_+,3} & \\
 & & q_{4,3_-} & & & q_{4_-,3} & * & q_{4_+,3} \\
 & & & q_{5,3_-} & & & q_{5_-,3} & *
\end{array}\right)$$

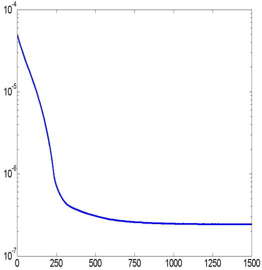
Basic theoretical result

Theorem

The Matrix $\begin{pmatrix} L_h & \mathbb{1} \\ \mathbb{1}^T & 0 \end{pmatrix}$ *is nonsingular.*

Function decrease

(M8B18N72.avi)



Benefit of the topological expansion

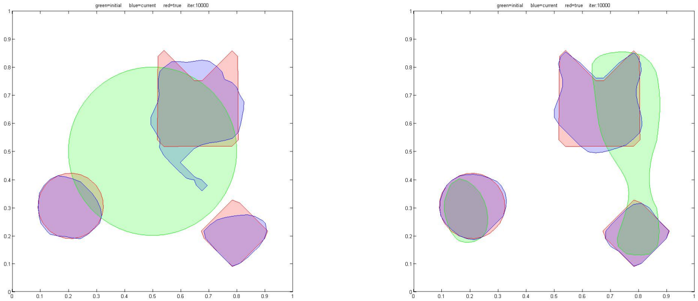


Figure: M=24, β = 1e-8

Dependence on regularization

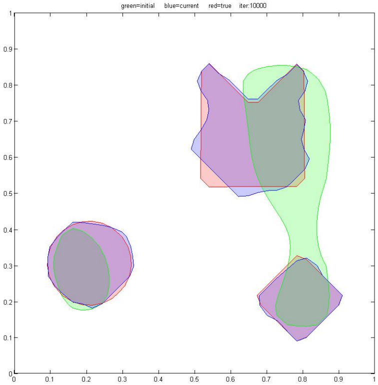


Figure: M=24, β = 1e-9

Dependence on regularization

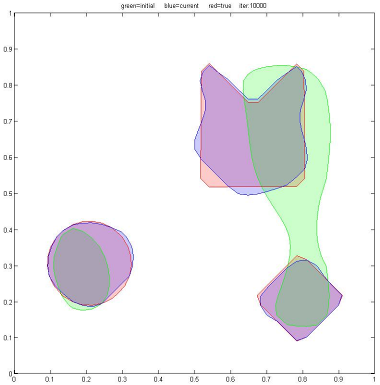


Figure: M=24, β = 1e-8

Dependence on regularization

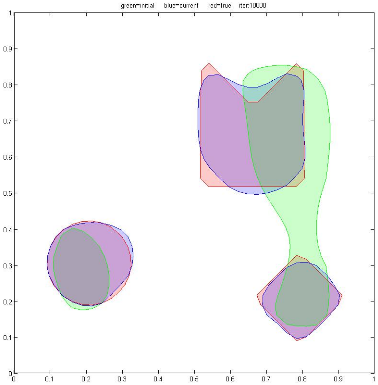


Figure: M=24, β = 1e-7

Dependence on regularization

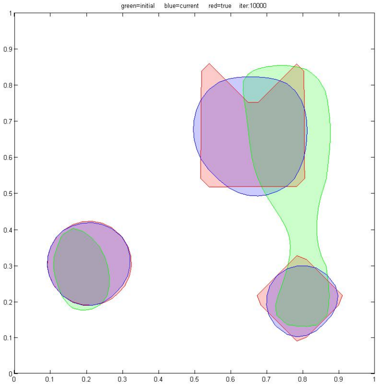
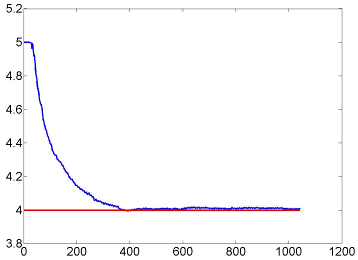
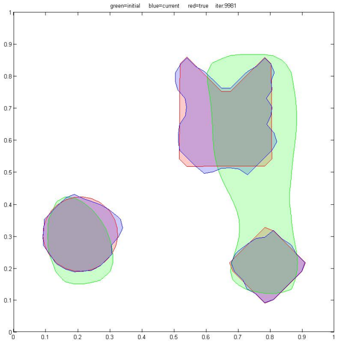


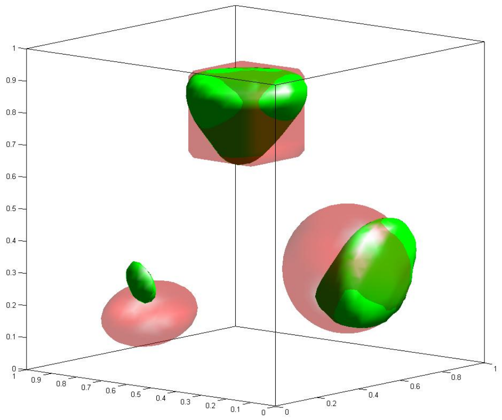
Figure: M=24, β = 1e-6

Different q_1





Topological expansion



Gradient method