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OPTIMAL CONTROL OF A SEMI-DISCRETE CAHN-HILLIARD-NAVIER-STOKES SYSTEM

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Abstract. In this paper, the optimal boundary control of a time-discrete Cahn-Hilliard Navier-Stokes system is studied. A general class of free energy potentials is considered which, in particular, includes the double-obstacle potential. The latter homogeneous free energy density yields an optimal control problem for a family of coupled systems, which result from a time discretization of a variational inequality of fourth-order and the Navier-Stokes equation. The existence of an optimal solution to the time discrete control problem as well as an approximate version is established. The latter approximation is obtained by mollifying the Moreau-Yosida approximation of the double obstacle potential. First order optimality conditions for the mollified problems are given, and besides the convergence of optimal controls of the mollified problems to an optimal control of the original problem, also first order optimality conditions for the original problem are derived through a limit process. The newly derived stationarity system is related to a function space version of C-stationarity.

Key words. Cahn-Hilliard/Navier-Stokes system, double-obstacle potential, mathematical programming with equilibrium constraints, optimal boundary control, Yosida regularization, C-stationarity.

AMS subject classifications. 35J87, 49J20, 34G25

1. Introduction.

The coupled Cahn-Hilliard/Navier-Stokes (CH-NS) system is a quantitative model which describes the hydrodynamics, such as demixing or phase separation, of multiphase-fluids. While the Navier-Stokes part of such a system captures the fluid dynamics over time (see, e.g., [13]), the Cahn-Hilliard model is related to a H^{-1}-gradient-flow for a Ginzburg-Landau free energy, which covers the phase separation behavior [12]. Mathematically, a CH-NS system describing the hydrodynamics of a two-phase fluid flow is given by

$$\partial_t v - \frac{1}{Re}\Delta v + v\cdot\nabla v + \nabla\pi + Kc\nabla w = 0 \text{ in } \Omega_T, \tag{1.1}$$
$$\text{div } v = 0 \qquad \text{in } \Omega_T, \tag{1.2}$$
$$\partial_t c - \frac{1}{Pe}\nabla\cdot(b(c)\nabla w) + v\cdot\nabla c = 0 \qquad \text{in } \Omega_T, \tag{1.3}$$
$$w \in \partial\Phi(c) - \gamma^2\Delta c \qquad \text{in } \Omega_T, \tag{1.4}$$
$$c(0) = c_a, \quad v(0) = v_a \qquad \text{in } \Omega \text{ at } t = 0, \tag{1.5}$$
$$\nabla c\cdot\vec{n} = 0, \quad \nabla w\cdot\vec{n} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{1.6}$$
$$v = r \qquad \text{on } \partial\Omega\times(0,T), \tag{1.7}$$

Here, v denotes the velocity of the fluid and π the related pressure, c, typically with values in $[-1,1]$, is the order parameter describing the mass concentrations c_1 and c_2 of the fluid phases and w is the associated chemical potential. The capillary number K, the Reynolds number Re, the Péclet number Pe, and the diffusivity parameter γ are given positive constants depending on material properties. The function $b(\cdot)$ represents the mobility involved in the phase separation process. Further, the space-time cylinder is given by $\Omega_T := \Omega\times(0,T)$, i.e., it is the Cartesian product of a spatial

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domain $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, and a time interval $[0,T]$ with $T > 0$ given. By $\partial\Omega$ we denote the boundary of Ω and by $\vec{n}$ the outward unit normal on $\partial\Omega$. By r we denote some prescribed boundary velocity, and c_a and v_a are given initial data.

In the above system, which is related to 'Model H' in [22], the mapping Φ corresponds to the homogeneous free energy density contained in the Ginzburg-Landau energy model. It is usually non-convex for capturing spinodal decomposition. Popular physically relevant choices are the logarithmic potential, which can be found, e.g., in Cahn's and Hilliard's seminal paper [12] but also in the Flory-Huggins theory for describing phase separation processes in the thermodynamics of polymer solutions, and the double obstacle potential [5,6,16], which, in the context of polymer solutions, appears appropriate to model situations of rapid wall-hardening [30]. A frequently used, but possibly less relevant choice for Φ in material science, is given by the double-well potential [14]. While the logarithmic and the double-well potentials enjoy differentiability properties (allowing to replace $\partial\Phi$ by the Fréchet derivative Φ'), respectively, the double obstacle potential has a possibly genuinely set-valued derivative $\partial\Phi(c)$ at c. The latter clearly complicates the situation, both analytically as well as numerically, and gives rise to a variational inequality in (1.4).

In this paper we are interested in the optimal control of the coupled CH-NS system. In this context, an objective functional $\mathcal{J}$ is minimized subject the CH-NS system, i.e., we seek to solve the problem

$$\text{minimize } \mathcal{J}(c,v,u) \text{ subject to (1.1)–(1.6)}, v = u \text{ on } \partial\Omega \times (0,T), \tag{1.8}$$

where the control u is an element of a closed, linear control space U. A particular instance of U is the closed, linear subspace of the trace space containing controls operating in the direction normal to a non-empty subset Γ_c of the boundary $\partial\Omega$ of the spatial domain Ω, only. We also refer to Problem 3.2 below for a time-discrete version of the optimization task (1.8).

With respect to applications the study of the above optimization problem is relevant for instance in the formation of polymeric membranes in the context of an immersion precipitation process. In this context, a polymer solution is immersed into a coagulation bath which contains a nonsolvent. Due to the concentration difference between the polymeric solution and the nonsolvent, the polymeric solution decomposes into two phases, a polymer-rich and a polymer-poor one, respectively. It is well-known [35] that the performance of the resulting polymer membrane depends significantly on its morphology (i.e., the porosity structure), which is the result of the phase separation process.

Optimal control problems for phase separation modelled either by the Cahn-Hilliard or the Allen-Cahn system were previously studied in [15,17,21,33,34]. In these papers, however, no coupling with other physically relevant systems occur. Concerning research on the coupled CH-NS system we mention that while some work on the analysis and numerics for the coupled CH-NS system is available [1, 7–10, 23–28], to the best of our knowledge the literature on the optimal control of the CH-NS system is essentially void. Hence, as a first step towards the optimal control of the CH-NS system with a rather general choice of the free energy, in this paper we study the optimal control of the time-discrete version of (1.8). We further note that semi-discretization in time is a common approach towards the numerical solution of time-dependent optimal control problems.

The rest of the paper is organized as follows: In section 2 the time-discrete CH-NS system is stated and an appropriate solution concept is introduced. Further, energy estimates are derived which are then used to prove existence of a solution of the time-discrete CH-NS system for a given control action u. With respect to the choice of the free energy, smooth as well as non-smooth homogeneous free energy densities are possible. This covers in particular the case of the double obstacle potential giving rise to a variational inequality. The associated semi-discrete optimal control problem is studied in section 3, where, besides existence of an optimal solution, a first order optimality system for a smooth free energy is derived. The latter includes the case of a mollified version of the Moreau-Yosida approximation of the double obstacle potential. Finally, in section 4 a stationarity system for the double obstacle potential is derived through a limit process of the associated stationarity system of section 3. The resulting system is of so-called C-stationarity type and is suitable for numerical realization.

1.1. Notation. In order to simplify the notation and to ease the exposition, from now on we set $Re = Pe = K = 1$, and we consider the constant mobility case only, i.e., $b(\cdot) \equiv 1$ in (1.1)–(1.6).

Further, $\mathbb{N}$ denotes the positive integers, $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, the extended natural and real numbers, respectively. The duality pairing between a Banach space X and its dual X^* is written as $\langle .,. \rangle_X : X^* \times X \to \mathbb{R}$. As usual, strong and weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively. For a Hilbert space H its inner product is given by $(.|.)_H : H \times H \to \mathbb{R}$, whereas $J_H : H \to H^*$ denotes the canonic isomorphism due to the Riesz theorem. Let $N \in \{2, 3\}$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. We define $W_0 := \{c \in L^2(\Omega) : \int_\Omega c = 0\}$, $W_1 := H^1(\Omega) \cap W_0$ with the norm $\|c\|_{W_1} := \|\nabla c\|_{L^2(\Omega)}$ and $W_{-1} := W_1^*$. Moreover, we set $V_1 := \{v \in H^1(\Omega; \mathbb{R}^N) : \operatorname{div} v = 0\}$, $\widetilde{V}_1 := V_1 \cap H_0^1(\Omega; \mathbb{R}^N)$, $V_0 := \widetilde{V}_0$ the closure of V_1 in $L^2(\Omega; \mathbb{R}^N)$, $V_{-1} := V_1^*$, $\widetilde{V}_{-1} := \widetilde{V}_1^*$ and $Y_i := W_i \times V_i$, $\widetilde{Y}_i := W_i \times \widetilde{V}_i$ for $i \in \{-1, 0, 1\}$. On $\widetilde{V}_1$ we use the norm $\|v\|_{\widetilde{V}_1}$ given by the functional $|\widetilde{v}|_{\widetilde{V}_1} := \|\nabla \widetilde{v}\|_{V_0}$ that is a seminorm on V_1. The other spaces are equipped with their standard norms. The space W_1^M denotes the M-times product $W_1 \times ... \times W_1$. Other product spaces are denoted analogously. We define $C_P := \|\operatorname{Id}\|_{\mathcal{L}(\widetilde{V}_1; V_1)}$. The subspace S_1 of $S := H^{1/2}(\partial\Omega; \mathbb{R}^N)$ is given by $S_1 := \operatorname{Tr}(V_1)$ where Tr denotes the usual zero-order trace operator $\operatorname{Tr} : H^1(\Omega; \mathbb{R}^N) \to H^{1/2}(\partial\Omega; \mathbb{R}^N)$. Moreover, $S_{-1} := S_1^*$.

For an arbitrary vector space X, $M \in \mathbb{N}$ and $\tau > 0$ we define the operators $D^+, \overline{\cdot}, S_+, S_- : X^{M+1} \to X^M$ by

$$\begin{aligned}
D^+(x_0, ..., x_M) &:= \tfrac{1}{\tau}(x_1 - x_0, ..., x_M - x_{M-1}),\\
\overline{(x_0, ..., x_M)} &:= \tfrac{1}{2}(x_0 + x_1, ..., x_{M-1} + x_M),\\
S_+(x_0, ..., x_M) &:= (x_1, ..., x_M),\\
S_-(x_0, ..., x_M) &:= (x_0, ..., x_{M-1}).
\end{aligned}$$

For the ease of notation, we use the following convention: Whenever we add $x \in X^{M_1}$ to $y \in X^{M_2}$ for $M_2 > M_1$, then this is understood in the sense $(x_0 + y_0, ..., x_{M_1-1} + y_{M_1-1})$, i.e. we project y onto its first M_1 components.

Remark 1.1. *For the proof of existence of a solution to the semidiscrete CH-NS system below, we reduce the inhomogeneous, discretized version of the Navier-Stokes equation (2.3) below to an equation satisfying homogeneous Dirichlet boundary conditions. This will be done with the help of the operator $F \in \mathcal{L}(S_1; V_1)$ satisfying $Tr \circ F = \operatorname{Id}_{S_1}$, where Id_{S_1} denotes the identity operator on S_1. Observe that such an operator F always exists. Indeed, by a result due to Heron [18], the subspace S_1 is* *given by* $\{u \in S : \int_{\partial\Omega} u \cdot \vec{n} = 0\}$. *In particular,* S_1 *is a Hilbert space and the trace operator regarded from* V_1 *into* S_1 *is a linear, bounded and surjective mapping between Hilbert spaces. Consequently, there exists a right inverse operator* $F \in \mathcal{L}(S_1; V_1)$ *to* Tr, *i.e.* $Tr \circ F = \mathrm{Id}_{S_1}$ *(cf. Aubin [2]). Moreover, we henceforth use the notation* F *for a right inverse as defined above.*

2. The semidiscrete CH-NS system. In our analysis we rely on the following discretization schemes and the pertinent notion of solution. For the corresponding mathematical description, below we use $Y_{-1}^M = Y_{-1} \times ... \times Y_{-1}$ (M-times) and analogously for the other relevant spaces. Further, the actions of operators like Δ and ∇ are understood in the componentwise sense whenever applied to elements of W_1^M and V_1^M, respectively, and analogously for the multi-valued operator A defined below.

Definition 2.1 (Solution of the semidiscrete CH-NS system). *Let* $M \in \mathbb{N}$, *initial data* $(c_a, v_a) \in Y_1$, *a right hand side* $(f^c, f^v) \in \widetilde{Y}_{-1}^M$, *a boundary value* $r \in S_1^{M+1}$ *and a multi-valued operator* $A \subset V_1 \times V_{-1}$ *be given. A pair* $(c, v) \in Y_1^{M+1}$ *is called a solution to the semidiscrete CH-NS system with data* $\mathcal{I} := (M, \tau, (c_a, v_a), (f^c, f^v), r, A)$, *if there exists* $w \in W_1^{M+1}$ *such that* $Tr\, v = r \in S_1^{M+1}$, $c_0 = c_a$, $w_0 = 0$, $v_0 = v_a$ *and*

$$D^+c - \Delta S_+ w + \nabla S_- c \cdot \overline{v} = f^c \quad \in W_{-1}^M, \tag{2.1}$$

$$-S_+ w - \Delta \overline{c} - I\overline{c} + AS_+ c \ni 0 \quad \in W_{-1}^M, \tag{2.2}$$

$$D^+ v - \Delta \overline{v} + (S_- v \cdot \nabla)\overline{v} - S_+ w \nabla S_- c = f^v \quad \in \widetilde{V}_{-1}^M. \tag{2.3}$$

In the definition above and depending on the context we consider the negative Laplace operator $-\Delta$ incorporating respective boundary conditions either as a mapping from W_1 into its dual W_{-1} or from V_1 into $\widetilde{V}_{-1}$. Similarly, I represents the canonical injection either of W_1 into W_{-1} or V_1 into $\widetilde{V}_{-1}$ again depending on the context. The operators $-\Delta : W_1 \to W_{-1}$ and $I : W_1 \to W_{-1}$ are self-adjoint whereas the adjoints of $-\Delta : V_1 \to \widetilde{V}_{-1}$ and $I : V_1 \to \widetilde{V}_{-1}$ read as follows:

$$-\Delta^* : \widetilde{V}_1 \to V_{-1}, \quad \langle -\Delta^* \widetilde{v}, v\rangle_{V_1} := \langle -\Delta v, \widetilde{v}\rangle_{\widetilde{V}_1},$$
$$I^* : \widetilde{V}_1 \to V_{-1}, \quad \langle I^* \widetilde{v}, v\rangle_{V_1} := \langle v, \widetilde{v}\rangle_{\widetilde{V}_1}.$$

The evaluation of the viscosity part $(u \cdot \nabla)v$ in q for $u, v, q \in V_1$ leads to the trilinear form $b(u, v, q)$, which, depending on the context, will be written in the following ways:

$$\langle (u \cdot \nabla)v, q\rangle_{V_1} := \langle b_1(v, q), u\rangle_{V_1} := \langle b_2(u, q), v\rangle_{V_1} := b(u, v, q) := \sum_{i,j=1}^{N} \int_\Omega u_i(\partial_i v_j) q_j.$$

It is well known that $\langle (u \cdot \nabla)\widetilde{v}, \widetilde{v}\rangle_{V_1} = 0$ for all $u \in V_1$ and $\widetilde{v} \in \widetilde{V}_1$; see, e.g., [32, Chapter 2, Lemma 1.3].

Remark 2.1. *Assume that* (c, v) *is a solution to the semidiscrete CH-NS system with data* $(M, \tau, (c_a, v_a), (f^c, f^v), r, A)$. *Then it holds that*

$$S_+ w = (-\Delta)^{-1}(f^c - D^+ c - \nabla S_- c \cdot \overline{v}) \tag{2.4}$$

where $(-\Delta)^{-1}$ *denotes the associated solution operator. Thus, the vector* w *is unique. Moreover, note that the assumption* $Tr\, v = r$ *implies* $r_0 = Tr\, v_a$. *This condition invokes compatability of the bounded-value* r *at initial time and the initial value* v_a *of the fluid velocity.*

The next reformulation of the problem proves to be useful for the following existence result.

LEMMA 2.2. *The system (2.1)–(2.3) is equivalent to the system*

$$\begin{aligned}
&(-\Delta)^{-1}D^+c + (-\Delta - I)\overline{c} + AS_+c \\
&\qquad +(-\Delta)^{-1}\nabla S_-c\cdot\overline{v} - (-\Delta)^{-1}f^c \ni 0 \quad \in W_{-1}^M, && (2.5)\\
&-S_+w + (-\Delta)^{-1}\big[f^c - (D^+c + \nabla S_-c\cdot\overline{v})\big] = 0 \quad \in W_1^M, && (2.6)\\
&D^+v - \Delta\overline{v} + (S_-v\cdot\nabla)\overline{v} - S_+w\nabla S_-c - f^v = 0 \quad \in \widetilde{V}_{-1}^M. && (2.7)
\end{aligned}$$

Proof. This result is easily seen by applying $(-\Delta)^{-1}$ to (2.1). □

DEFINITION 2.3. *We define the solution sets $\mathcal{S}(\mathcal{I}) \subset Y_1^{M+1}$ and $\mathcal{S}^w(\mathcal{I}) \subset (W_1 \times W_1 \times V_1)^{M+1}$, respectively, as*

$$\begin{aligned}
\mathcal{S}(\mathcal{I}) &:= \{(c,v) : (c,v) \textit{ is a solution to the semidiscrete CH-NS system for}\\
&\qquad \textit{given data } \mathcal{I} = (M,\tau,(c_a,v_a),(f^c,f^v),r,A)\},\\
\mathcal{S}^w(\mathcal{I}) &:= \{(c,w,v) : (c,v)\in\mathcal{S}(\mathcal{I}),\ w_0 = 0,\ S_+w \textit{ satisfies } (2.4)\}.
\end{aligned}$$

2.1. Energy estimates I. Before establishing the existence of solutions to the semidiscrete CH-NS system we study energy estimates. Such estimates are useful in the existence proof in order to show that solutions to suitable auxilary problems yield solutions to the original problem.

DEFINITION 2.4 (Energy functionals). *For a given potential $\varphi : W_1 \to \overline{\mathbb{R}}$ and for $(c,v) \in Y_1$ we define the free energy, the kinectic energy and the (total) energy, respectively, according to*

$$\mathrm{E}_{\mathrm{free}}(c) := \frac{1}{2}\big[\|c\|_{W_1}^2 - \|c\|_{W_0}^2\big] + \varphi(c), \qquad \mathrm{E}_{\mathrm{kin}}(v) := \frac{1}{2}\|v\|_{V_0}^2,$$
$$\mathrm{E}(c,v) := \mathrm{E}_{\mathrm{free}}(c) + \mathrm{E}_{\mathrm{kin}}(v).$$

Here and below for a convex functional φ, the operator $\partial\varphi$ denotes the subdifferential of convex analysis and $D(A)$ denotes the domain of a given operator A.

LEMMA 2.5. *Assume that $\tau > 0$, $(c,\widetilde{v}) \in Y_1^2$, $w_1 \in W_1$, $u \in V_1$, $(f^c,f^v) \in \widetilde{Y}_{-1}$, $\varphi : V_1 \to \overline{\mathbb{R}}$ convex, $\varphi(c_a) < \infty$, $A = \partial\varphi \subset W_1 \times W_{-1}$ and*

$$D^+c - \Delta w_1 = f^c, \quad w_1 \in -\Delta\overline{c} - I\overline{c} + AS_+c, \quad D^+\widetilde{v} - \Delta\overline{\widetilde{v}} + (u\cdot\nabla)\overline{\widetilde{v}} = f^v. \quad (2.8)$$

Then for $c = (c_0,c_1)$ and $\widetilde{v} = (\widetilde{v}_0,\widetilde{v}_1)$ it holds that

$$\mathrm{E}_{\mathrm{free}}(c_1) - \mathrm{E}_{\mathrm{free}}(c_0) \le \tau\big[\langle f^c, w_1\rangle_{W_1} - \|w_1\|_{W_1}^2\big],$$
$$\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0) \le \tau\Big[\Big\langle f^v, \overline{\widetilde{v}}\Big\rangle_{\widetilde{V}_1} - \|\overline{\widetilde{v}}\|_{\widetilde{V}_1}^2\Big].$$

Proof. The inclusion for w_1 implies $c_1 \in D(A)$ and therefore $\varphi(c_1) < \infty$. We set $v^* := w_1 - (-\Delta\overline{c} - I\overline{c})$, hence $v^* \in Ac_1 = \partial\varphi(c_1)$. Using the latter equation, (2.8) and the definition of the free energy and the given equation, we find that

$$
\begin{aligned}
&2\big(\mathrm{E}_{\mathrm{free}}(c_1)-\mathrm{E}_{\mathrm{free}}(c_0)\big)\\
&\quad=\big[\|c_1\|_{W_1}^2-\|c_0\|_{W_1}^2\big]-\big[\|c_1\|_{W_0}^2-\|c_0\|_{W_0}^2\big]+2\big[\varphi(c_1)-\varphi(c_0)\big]\\
&\quad=(\nabla(c_1+c_0)|\nabla(c_1-c_0))_{W_0}-(c_1+c_0|c_1-c_0)_{W_0}+2\big[\varphi(c_1)-\varphi(c_0)\big]\\
&\quad\le\langle-\Delta(c_1+c_0),c_1-c_0\rangle_{W_1}+\langle-I(c_1+c_0),c_1-c_0\rangle_{W_1}+2\,\langle v^*,c_1-c_0\rangle_{W_1}\\
&\quad=2\tau\left\langle-\Delta\overline{c}-I\overline{c}+v^*,D^+c\right\rangle_{W_1}\;=\;2\tau\left\langle w_1,D^+c\right\rangle_{W_1}\;=\;2\tau\left\langle D^+c,w_1\right\rangle_{W_1}\\
&\quad=2\tau\,\langle f^c+\Delta w_1,w_1\rangle_{W_1}\;=\;2\tau\big[\,\langle f^c,w_1\rangle_{W_1}-\|w_1\|_{W_1}^2\big].
\end{aligned}
$$

Analogously, it follows that

$$
\begin{aligned}
&2\big(\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1)-\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0)\big)\\
&\quad=\|\widetilde{v}_1\|_{V_0}^2-\|\widetilde{v}_0\|_{V_0}^2\;=\;(\widetilde{v}_1-\widetilde{v}_0|\widetilde{v}_1+\widetilde{v}_0)_{V_0}\;=\;\langle\widetilde{v}_1-\widetilde{v}_0,\widetilde{v}_1+\widetilde{v}_0\rangle_{\widetilde{V}_1}\\
&\quad=2\tau\left\langle D^+\widetilde{v},\overline{\widetilde{v}}\right\rangle_{\widetilde{V}_1}\;=\;2\tau\left\langle f^v+\Delta\overline{\widetilde{v}}-(u\cdot\nabla)\overline{\widetilde{v}},\overline{\widetilde{v}}\right\rangle_{\widetilde{V}_1}\;=\;2\tau\big[\left\langle f^v,\overline{\widetilde{v}}\right\rangle_{\widetilde{V}_1}-\|\overline{\widetilde{v}}\|_{\widetilde{V}_1}^2\big],
\end{aligned}
$$

which completes the proof. ∎

Next we provide an estimate for the total energy related to a modified version of our original problem (2.1)–(2.3) which allows a specific nonlinear coupling.

LEMMA 2.6. *Let* $\tau>0$, $\varphi:W_1\to\overline{\mathbb{R}}$ *be convex,* $A=\partial\varphi$, $(c,w,v)\in W_1^2\times W_1^2\times V_1^2$, $(f^c,f^v)\in\widetilde{Y}_{-1}$, $r\in S_1^2$ *and* $\varphi(c_0)<\infty$. *Moreover, assume we are given operators* $B\in\mathcal{L}(V_1;\widetilde{V}_{-1})$, $Q=(Q_1,Q_2):\widetilde{Y}_1\to\widetilde{Y}_{-1}$ *such that for all* $\hat{v}\in\widetilde{V}_1$, $\hat{w}\in W_1$

$$
\langle B\hat{v},\hat{v}\rangle_{\widetilde{V}_1}=0,\qquad\langle Q(\hat{w},\hat{v}),(\hat{w},\hat{v})\rangle_{\widetilde{Y}_1}=0.
$$

For $\widetilde{v}:=v-Fr\in\widetilde{V}_1^2$ *(with an implicit requirement for* v*) assume that*

$$
D^+c-\Delta w_1+Q_1(w_1,\overline{\widetilde{v}})+\nabla c_0\cdot F\overline{r}=f^c,\quad w_1\in-\Delta\overline{c}-I\overline{c}+AS_+c,\tag{2.9}
$$
$$
D^+v-\Delta\overline{v}+B\overline{v}+Q_2(w_1,\overline{\widetilde{v}})=f^v.\tag{2.10}
$$

Then for a constant C *depending only on* τ, Ω *and* $\|Fr\|_{V_1^2}$, *with* $z_1:=f^c-\nabla c_0\cdot F\overline{r}$ *and* $z_2:=f^v-[FD^+r-\Delta F\overline{r}+BF\overline{r}]$, *it holds that*

$$
\mathrm{E}(c_1,v_1)-\mathrm{E}(c_0,v_0)+\frac{\tau}{2}\big[\|w_1\|_{W_1}^2+\|\overline{\widetilde{v}}\|_{\widetilde{V}_1}^2\big]\;\le\;C(\|z_1\|_{W_{-1}}^2+\|z_2\|_{\widetilde{V}_{-1}}^2+\|\widetilde{v}_0\|_{V_0}^2+1).
$$

Proof. Since $\overline{\widetilde{v}}\in\widetilde{V}_1$ satisfies

$$
D^+\widetilde{v}-\Delta\overline{\widetilde{v}}+B\overline{\widetilde{v}}=z_2-Q_2(w_1,\overline{\widetilde{v}}),
$$

it follows from Lemma 2.5 that

$$
\begin{aligned}
\mathrm{E}_{\mathrm{free}}(c_1)-\mathrm{E}_{\mathrm{free}}(c_0)\;&\le\;\tau\big[\,\langle z_1,w_1\rangle_{W_1}-\left\langle Q_1(w_1,\overline{\widetilde{v}}),w_1\right\rangle_{W_1}-\|w_1\|_{W_1}^2\big],\\
\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1)-\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0)\;&=\;\tau\big[\left\langle z_2,\overline{\widetilde{v}}\right\rangle_{\widetilde{V}_1}-\left\langle Q_2(w_1,\overline{\widetilde{v}}),\overline{\widetilde{v}}\right\rangle_{\widetilde{Y}_1}-\|\overline{\widetilde{v}}\|_{\widetilde{V}_1}^2\big].
\end{aligned}
$$

In order to pass from $\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_i)$ to $\mathrm{E}_{\mathrm{kin}}(v_i)$ we use

$$
\begin{aligned}
&2\big[\mathrm{E}_{\mathrm{kin}}(v_1) - \mathrm{E}_{\mathrm{kin}}(v_0)\big] \\
&\quad = 2\big[\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1 + Fr_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0 + Fr_0)\big] \;=\; \|\widetilde{v}_1 + Fr_1\|_{V_0}^2 - \|\widetilde{v}_0 + Fr_0\|_{V_0}^2 \\
&\quad = (\widetilde{v}_1 + \widetilde{v}_0 + F(r_0 + r_1) | \widetilde{v}_1 - \widetilde{v}_0 + F(r_1 - r_0))_{V_0} \\
&\quad \le \|\widetilde{v}_1\|_{V_0}^2 - \|\widetilde{v}_0\|_{V_0}^2 + (2\|\overline{\widetilde{v}}\|_{V_0} + \|\widetilde{v}_1 - \widetilde{v}_0\|_{V_0})(\|Fr_0\|_{V_0} + \|Fr_1\|_{V_0}) \\
&\qquad + (\|Fr_0\|_{V_0} + \|Fr_1\|_{V_0})^2 \\
&\quad \le 2\big[\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0)\big] + C(4\|\overline{\widetilde{v}}\|_{V_0} + 2\|\widetilde{v}_0\|_{V_0} + 1) \\
&\quad \le 2\big[\mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0)\big] + \tfrac{\tau}{2}\|\overline{\widetilde{v}}\|_{V_1}^2 + C(\|\widetilde{v}_0\|_{V_0}^2 + 1)
\end{aligned}
$$

for constants C depending only on Ω, τ and $\|Fr\|_{V_0^2}$. From this relation we obtain the estimate

$$
\begin{aligned}
&\mathrm{E}(c_1, v_1) - \mathrm{E}(c_0, v_0) \\
&\quad \le \mathrm{E}_{\mathrm{free}}(c_1) - \mathrm{E}_{\mathrm{free}}(c_0) + \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_1) - \mathrm{E}_{\mathrm{kin}}(\widetilde{v}_0) + \tfrac{\tau}{4}\|\overline{\widetilde{v}}\|_{V_1}^2 + C(\|\widetilde{v}_0\|_{V_0}^2 + 1) \\
&\quad \le \tau\Big[\langle z_1, w_1\rangle_{W_1} + \Big\langle z_2, \overline{\widetilde{v}}\Big\rangle_{\widetilde{V}_1} - \|w_1\|_{W_1}^2 - \tfrac{3}{4}\|\overline{\widetilde{v}}\|_{\widetilde{V}_1}^2\Big] + C(\|\widetilde{v}_0\|_{V_0}^2 + 1) \\
&\quad \le -\tfrac{\tau}{2}\Big[\|w_1\|_{W_1}^2 + \|\overline{\widetilde{v}}\|_{\widetilde{V}_1}^2\Big] + C(\|z_1\|_{W_{-1}}^2 + \|z_2\|_{\widetilde{V}_{-1}}^2 + \|\widetilde{v}_0\|_{V_0}^2 + 1).
\end{aligned}
$$

This yields the assertion. ☐

Corollary 2.7. *Let the assumptions of Lemma 2.6 be satisfied. If the functional* $\varphi : W_1 \to \mathbb{R}$ *satisfies*

$$
(\mathrm{H}_1) \qquad \varphi(f) - \tfrac{1}{2}\|f\|_{W_0}^2 \;\ge\; C_\varphi
$$

for a constant $C_\varphi \in \mathbb{R}$ *and every* $f \in W_1$, *then there exists a constant* $m > 0$ *depending on* $\tau, \Omega, c_a, v_a, \varphi(c_a), C_\varphi$ *and* Fr *such that every solution* (c, w, v) *to (2.9)–(2.10) satisfies*

$$
\|w_1\|_{W_1}^2 + \|\frac{1}{2}(\widetilde{v}_i + \widetilde{v}_{i+1})\|_{\widetilde{V}_1}^2 \;\le\; m^2.
$$

Proof. This is a direct consequence of Lemma 2.6 and $\mathrm{E}(c_1, v_1) \ge C_\varphi$. ☐

2.2. Existence of solutions to the semidiscrete CH-NS system. The existence of a solution to the semidiscrete CH-NS system for one time step is studied next. For an arbitrary finite number of steps M, this result will be applied iteratively in the proof of Theorem 2.13 below.

In the existence proof we utilize results on several classes of operators. For the reader's convenience we briefly recall the definitions of these classes. A multi-valued operator $A \subset X \times X^*$ mapping a Banach space X into its dual is called *(strongly) monotone* if there exists a constant $\alpha \ge 0$ ($\alpha > 0$) such that

$$
\langle x_1^* - x_2^*, x_1 - x_2\rangle_X \ge \alpha\|x_1 - x_2\|_X^2
$$

for all $(x_1, x_1^*), (x_2, x_2^*) \in A$, and it is *maximal monotone* if it is maximal among all monotone operators. A single-valued operator $A : X \to X^*$ is *pseudomonotone* if and only if for every sequence (x_n) in X with $x_n \rightharpoonup x$ the implication

$$
\overline{\lim}\,\langle Ax_n, x_n - x\rangle \le 0 \qquad \Longrightarrow \qquad \langle Ax, x - v\rangle_X \le \underline{\lim}\,\langle Ax_n, x_n - v\rangle_X
$$

is satisfied for every $v \in X$. Here and below, $\underline{\lim}$ and $\overline{\lim}$ denote the limit inferior and the limit superior, respectively. Finally, $A : X \to X^*$ is said to be *totally continuous* if $x_n \rightharpoonup x$ in X implies $Ax_n \to Ax$ in X^*. Thus, every totally continuous operator is pseudomonotone.

PROPOSITION 2.8. *Let $(c_a, v_a) \in Y_1$, $(\hat{g}^c, \hat{g}^v) \in \widetilde{Y}_{-1}$ and $r \in S_1^2$ with $Tr\, v_a = r_0$. Assume that $A \subset W_1 \times W_{-1}$ is maximal monotone, and $R = (R_1, R_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ is pseudomonotone and bounded such that for some constant C_1 and all $(\hat{c}, \hat{v}) \in Y_1$ it holds that*

$$\langle R(\hat{c}, \hat{v}), (\hat{c}, \hat{v})\rangle_{Y_1} \geq -C_1(1 + \|\hat{c}\|_{W_1} + \|\hat{v}\|_{V_1}). \tag{2.11}$$

Moreover, suppose that $(\mathcal{A}_1 + \mathcal{B}^{v_a})|_{\widetilde{Y}_1} : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ is coercive for the two operators $\mathcal{A}_1, \mathcal{B}^{v_a} : Y_1 \to Y_{-1}$ defined for all $(\hat{c}, \hat{v}) \in Y_1$ by

$$\mathcal{A}_1 \hat{c} := \big((\tfrac{2}{\tau}(-\Delta)^{-1} - \Delta)\hat{c}, (\tfrac{2}{\tau} - \Delta)\hat{v} \big), \quad \mathcal{B}^{v_a}(\hat{c}, \hat{v}) := \big(- I\hat{c}, (v_a \cdot \nabla)\hat{v} \big).$$

Then there exists a pair $(c, v) \in Y_1^2$ such that

$$Tr\, v = r,\ c_0 = c_a,\ v_0 = v_a, \tag{2.12}$$
$$(-\Delta)^{-1} D^+ c + (-\Delta - I)\overline{c} + AS_+ c + R_1(\overline{c}, \overline{v} - F\overline{r}) \ni \hat{g}^c, \tag{2.13}$$
$$D^+ v - \Delta\overline{v} + (S_- v \cdot \nabla)\overline{v} + R_2(\overline{c}, \overline{v} - F\overline{r}) = \hat{g}^v. \tag{2.14}$$

Proof. Using $D^+ c = \frac{2}{\tau}(\overline{c} - S_- c)$, $S_+ c = 2\overline{c} - S_- c$ and analogue relations for v, the pair (c, v) solves (2.13)–(2.14) if and only if

$$(\tfrac{2}{\tau}(-\Delta)^{-1} - \Delta - I)\overline{c} + A(2\overline{c} - S_- c) + R_1(\overline{c}, \overline{v} - F\overline{r}) = \hat{g}^c + \tfrac{2}{\tau}(-\Delta)^{-1} S_- c,$$
$$(\tfrac{2}{\tau} - \Delta)\overline{v} + (S_- v \cdot \nabla)\overline{v} + R_2(\overline{c}, \overline{v} - F\overline{r}) = \hat{g}^v + \tfrac{2}{\tau} S_- v.$$

If we set $(\hat{c}, \hat{v}) := (\overline{c}, \overline{v} - F\overline{r}) \in Y_1$ and $(g^c, g^v) := \big(\hat{g}^c + \frac{2}{\tau}(-\Delta)^{-1} S_- c,\ \hat{g}^v + \frac{2}{\tau} S_- v - \big[(\frac{2}{\tau} - \Delta)F\overline{r} + (S_- v \cdot \nabla)F\overline{r}\big]\big)$ and define the operator $\mathcal{A}_2 \subset Y_1 \times Y_{-1}$ by

$$\mathcal{A}_2(\hat{c}, \hat{v}) := \{(A(2\hat{c} - S_- c), 0) : \hat{c} \in W_1\},$$

then $(c, v) \in Y_1^2$ is a solution of (2.12)–(2.14) if and only if $(\hat{c}, \hat{v}) \in Y_1$ satisfies $\hat{v} \in \widetilde{V}_1$ and

$$(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{B}^{v_a} + R)(\hat{c}, \hat{v}) = (g^c, g^v).$$

The operator $\mathcal{A}_1$ is strongly monotone, $\mathcal{A}_2$ is maximal monotone, and the operators $\mathcal{B}^{v_a}$ and R are pseudomonotone and bounded. Obviously, these properties remain valid if we restrict the domain of those operators and the domain of their images to $\widetilde{Y}_1$ and $\widetilde{Y}_{-1}$, respectively. Moreover, $\mathcal{A}_1 + \mathcal{B}^{v_a} + R$ is coercive on $\widetilde{Y}_1$ by our assumptions. Therefore, Browder's Theorem [11] implies that $\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{B}^{v_a} + R \subset \widetilde{Y}_1 \times \widetilde{Y}_{-1}$ is surjective. This finishes the proof. ◻

For the iterative application of this proposition to guarantee the existence of a solution to the semidiscrete CH-NS system several auxiliary results have to be established.

LEMMA 2.9. *Suppose that H is a real Hilbert space and J, L and K are linear, bounded operators on H such that J is monotone and injective, L is strongly monotone* *and K is compact. Moreover, J and K are symmetric and commute, i.e. $JK = KJ$. Then there exists an $\alpha \in \mathbb{R}$ such that $\alpha J + L + K$ is strongly monotone.*

Proof. The operator L is coercive. Hence there exists $\beta > 0$ such that $(Lx|x)_H \geq \beta\|x\|_H^2$ for all $x \in H$. Since K is compact and symmetric, the subspace H_1 given by the span of $\{x \in H \mid Kx = \lambda x \text{ for a } \lambda \in \mathbb{R} \text{ with } |\lambda| \geq \frac{\beta}{4}\}$ is finite dimensional and $\|Kz\|_H \leq \frac{\beta}{4}\|z\|_H$ for $z \in H_1^\perp \subset H$. Since J is monotone and injective, $\gamma := \inf\{(Jy|y)_H \mid y \in H_1,\ \|y\|_H = 1\} > 0$. Because J and K commute, J maps H_1 into itself. Indeed, given $x \in H$ with $Kx = \lambda x$ it follows that $K(Jx) = JKx = \lambda Jx$. Hence, for $x \in H_1 \subset H$ it follows that $Jx \in H_1$. Therefore, also $H_1^\perp$ is an invariant subspace under J. This follows from the fact that for $y \in H_1$, $z \in H_1^\perp$: $(Jz|y)_H = (z|Jy)_H = 0$ since $Jy \in H_1$.

Now, put $\alpha := \frac{1}{\gamma}(\|K\|_H + \frac{4}{\beta}\|K\|_H^2)$. Then for arbitrary $x \in H$ and for $y \in H_1, z \in H_1^\perp$ with $x = y + z$ we obtain:

$$
\begin{aligned}
(Lx|x)_H &\geq \beta\|x\|_H^2 = \beta\big[\|y\|_H^2 + \|z\|_H^2\big],\\
(Kx|x)_H &= (K(y+z)|y+z)_H\\
&\geq -\big[\|K\|_H\|y\|_H^2 + 2\|K\|_H\|y\|_H\|z\|_H + \tfrac{\beta}{4}\|z\|_H^2\big]\\
&\geq -\big[(\|K\|_H + \tfrac{4}{\beta}\|K\|_H^2)\|y\|_H^2 + \tfrac{\beta}{2}\|z\|_H^2\big],\\
(Jx|x)_H &= (J(y+z)|y+z)_H = (Jy|y)_H + (Jz|z)_H \geq \gamma\|y\|_H^2.
\end{aligned}
$$

Consequently, $((\alpha J + L + K)x|x)_H \geq \frac{\beta}{2}\big[\|y\|_H^2 + \|z\|_H^2\big] = \frac{\beta}{2}\|x\|_H^2$. ☐

Lemma 2.9 is useful in further characterizing the operator $(\mathcal{A}_1 + \mathcal{B}^{v_a})|_{\widetilde{Y}_1}$ of Proposition 2.8.

LEMMA 2.10. *There exists a constant $\tau_0 > 0$ depending only on Ω such that for every $0 < \tau \leq \tau_0$ and for the operators $\mathcal{A}_1$ and $\mathcal{B}^{v_a}$ given in Proposition 2.8, the sum $(\mathcal{A}_1 + \mathcal{B}^{v_a})|_{\widetilde{Y}_1} : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ is strongly monotone.*

Proof. Let us consider $H := W_1$ and the operators $J, L, K \in \mathcal{L}(H)$ that are given by

$$
\begin{gathered}
K := (-\Delta)^{-1}(-I) = -(-\Delta)^{-1}, \quad L := (-\Delta)^{-1}(-\Delta) = \mathrm{Id}_H,\\
J := (-\Delta)^{-1}(-\Delta)^{-1} = K^2.
\end{gathered}
$$

Since $I \in \mathcal{L}(W_1, W_{-1})$ is injective and compact, K and J have the same property as well. Moreover, they are also symmetric since

$$
(Kw_1|w_2)_H = -\langle w_1, w_2\rangle_H = -\langle w_2, w_1\rangle_H = (Kw_2|w_1)_H,
$$

they commute since $J = K^2$, and J is monotone because of $(Jw|w)_H = (Kw|Kw)_H \geq 0$. By Lemma 2.9 there exists an $\alpha > 0$ such that $\alpha J + L + K$ is coercive. Obviously, increasing α preserves this property. Now, for $\tau_0 := \frac{\alpha}{2}$ and $0 < \tau \leq \tau_0$ we obtain for every $(c, \tilde{v}) \in \widetilde{Y}_1$ by the definition of $\mathcal{A}_1$ and $\mathcal{B}^{v_a}$ and due to $\langle (v_a \cdot \nabla)\tilde{v}, \tilde{v}\rangle_{V_1} = 0$ that

$$
\begin{aligned}
\langle (\mathcal{A}_1 + \mathcal{B}^{v_a})(c,\tilde{v}), (c,\tilde{v})\rangle_{Y_1} &= \big\langle (\tfrac{2}{\tau}(-\Delta)^{-1} - \Delta - I)c, c\big\rangle_{W_1} + \big\langle (\tfrac{2}{\tau}I - \Delta)\tilde{v}, \tilde{v}\big\rangle_{V_1}\\
&\geq ((\alpha J + L + K)c|c)_{W_1} + \min(\tfrac{2}{\tau}, 1)\|\tilde{v}\|_{V_1}^2\\
&\geq \beta_1\|c\|_{W_1}^2 + \min(\alpha^{-1}, 1)\|\tilde{v}\|_{V_1}^2\\
&\geq \beta_2\|(c,\tilde{v})\|_{Y_1}^2
\end{aligned}
$$

for some $\beta_1, \beta_2 > 0$. This proves the assertion. ☐

Subsequently, the constant τ_0 of Lemma 2.10 will be denoted by $\tau_0(\Omega)$ in order to make the dependence on Ω explicit in the notation. In the following lemma we study properties of the terms coupling the Cahn-Hilliard and the Navier-Stokes systems.

LEMMA 2.11. *Let $c_a \in W_1$ be given. Then the operator $P = (P_1, P_2) : Y_1 \to \widetilde{Y}_{-1}$ defined by*

$$P_1(\hat{w}, \hat{v}) := \nabla c_a \cdot \hat{v}, \quad P_2(\hat{w}, \hat{v}) := -\hat{w}\nabla c_a$$

is bounded, bilinear, totally continuous and satisfies $\langle P(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{Y_1} = 0$ for $(\hat{w}, \hat{v}) \in \widetilde{Y}_1$.

Proof. By Sobolev's embedding theorem and since $N \leq 3$, the mappings $(\hat{c}, \hat{v}) \mapsto \nabla\hat{c}\cdot\hat{v} : W_1 \times V_1 \to W_{-1}$ and $(\hat{c}, \hat{w}) \mapsto \hat{w}\nabla\hat{c} : W_1 \times W_1 \to \widetilde{V}_{-1}$ are bilinear, bounded, and compact in both components. Thus, P is totally continuous, bounded and satisfies $\langle P(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{Y_1} = 0$. ∎

PROPOSITION 2.12. *Suppose we are given $0 < \tau \leq \tau_0(\Omega)$, $(c_a, v_a), (c_b, v_b) \in Y_1$, $(f^c, f^v) \in \widetilde{Y}_{-1}$, $r \in S_1^2$ with $Tr\, v_a = r_0$, a proper, convex and lower-semicontinuous functional $\varphi : W_1 \to \mathbb{R}$ satisfying* (H$_1$) *for a constant $C_\varphi \in \mathbb{R}$, and $A := \partial\varphi \subset W_1 \times W_{-1}$. Then there exists a triple $(c, w, v) \in (W_1 \times W_1 \times V_1)^2$ such that $c_0 = c_a$, $w_0 = 0$, $v_0 = v_a$, $Tr\, v = r$ and*

$$\begin{aligned} D^+c - \Delta w_1 + \nabla c_b \cdot \overline{v} &= f^c, \\ w_1 &\in (-\Delta - I)\overline{c} + AS_+c, \\ D^+v - \Delta\overline{v} + (v_b \cdot \nabla)\overline{v} - w_1\nabla c_b &= f^v. \end{aligned}$$

In particular, if $(c_a, v_a) = (c_b, v_b)$, then $(c, w, v) \in \mathcal{S}^w(1, \tau, (c_a, v_a), (f^c, f^v), r, A)$.

Proof. The proof is decomposed into several steps. First we show the boundedness of solutions independently of the involved operators followed by the precise construction of the associated operators. In step 3 we show the total continuity of the operator providing w given (c, v). The last two steps show applicability of Proposition 2.8 and the fact that we obtain a solution of the original problem by our proof technique.

1. Assume that we are given an operator $Q = (Q_1, Q_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ satisfying $\langle Q(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{\widetilde{Y}_1} = 0$ for all $(\hat{w}, \hat{v}) \in \widetilde{Y}_1$. By Corollary 2.7 there exists a constant $m > 0$ depending on $\tau, \Omega, Fr, v_a, f^c$ and f^v such that every solution $(c, w, v) \in (W_1 \times W_1 \times V_1)^2$, $\widetilde{v} := v - Fr \in \widetilde{V}_1$, to

$$\begin{gathered} \mathrm{Tr}\, v = r,\ c_0 = c_a,\ v_0 = v_a, \\ D^+c - \Delta w_1 + Q_1(w_1, \overline{\widetilde{v}}) + \nabla c \cdot v(c_b, F\overline{r}) = f^c, \quad w_1 \in -\Delta\overline{c} - I\overline{c} + AS_+c \\ D^+v - \Delta\overline{v} + (v_b \cdot \nabla)\overline{v} + Q_2(w_1, \overline{\widetilde{v}}) = f^v. \end{gathered}$$

satisfies the estimate

$$\|w_1\|_{W_1}^2 + \|\overline{\widetilde{v}}\|_{\widetilde{V}_1}^2 \leq m^2.$$

In particular, m does not depend on Q. Moreover, with $P_1(\hat{w}, \hat{v}) := \nabla c_b \cdot \hat{v}$ and $P_2(\hat{w}, \hat{v}) := -\hat{w}\nabla c_b$, according to Lemma 2.11 for $(\hat{w}, \hat{v}) \in \widetilde{Y}_1$ we have that

$$\begin{aligned} \|P_1(\hat{w}, \hat{v})\|_{W_{-1}} &= \|\nabla c_b \cdot \hat{v}\|_{W_{-1}} \leq \beta_1\|\hat{v}\|_{\widetilde{V}_1}, \\ \|P_2(\hat{w}, \hat{v})\|_{\widetilde{V}_{-1}} &= \|-\hat{w}\nabla c_b\|_{\widetilde{V}_{-1}} \leq \beta_2\|\hat{w}\|_{\widetilde{V}_1}, \end{aligned}$$

for constants $\beta_1, \beta_2 > 0$ depending on Ω and c_b. The operator $-\Delta : W_1 \to W_{-1}$ is strongly monotone with constant 1.

2. Let τ with $0 < \tau \le \tau_0(\Omega)$ and define a function $d : \mathbb{R} \to \mathbb{R}$ and a norm $|||.|||$ on $\widetilde{Y}_{-1}$ by

$$d(r) := \begin{cases} 1 & \text{if } r \le 1, \\ 2 - r & \text{if } 1 < r < 2, \\ 0 & \text{if } 2 \le r, \end{cases}$$

$$|||(w^*, v^*)||| := \frac{1}{2m}\left(\frac{1}{\beta_1}\|w^*\|_{W_{-1}} + \min\left(\frac{1}{\beta_2}, \frac{1}{4\beta_1\beta_2}\right)\|v^*\|_{\widetilde{V}_{-1}}\right).$$

Furthermore, we define the operators $Q = (Q_1, Q_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ and $Q_1^{\hat{v}}, M_{\hat{v}} : W_1 \to W_{-1}$ for $\hat{v} \in \widetilde{V}_1$ by

$$\begin{aligned} Q(\hat{w}, \hat{v}) &:= d\big(|||P(\hat{w}, \hat{v})|||\big) P(\hat{w}, \hat{v}), \\ Q_1^{\hat{v}} \hat{w} &:= Q_1(\hat{w}, \hat{v}), \\ M_{\hat{v}} \hat{w} &:= -\Delta \hat{w} + Q_1^{\hat{v}} \hat{w}. \end{aligned}$$

The operator Q inherits the property $\langle Q(\hat{w}, \hat{v}), (\hat{w}, \hat{v})\rangle_{\widetilde{Y}_1} = 0$ from P. For $\hat{w}_1, \hat{w}_2 \in W_1$ and $\hat{v} \in \widetilde{V}_1$ we have

$$\begin{aligned} \|Q_1^{\hat{v}}\hat{w}_1 - Q_1^{\hat{v}}\hat{w}_2\|_{W_{-1}} &= \|d\big(|||P(\hat{w}_1, \hat{v})|||\big)P_1(\hat{w}_1, \hat{v}) - d\big(|||P(\hat{w}_2, \hat{v})|||\big)P_1(\hat{w}_2, \hat{v})\|_{W_{-1}} \\ &= \big|d\big(|||P(\hat{w}_1, \hat{v})|||\big) - d\big(|||P(\hat{w}_2, \hat{v})|||\big)\big| \, \|P_1(\hat{w}_1, \hat{v})\|_{W_{-1}}, \end{aligned}$$

since $P_1(\hat{w}_1, \hat{v}) = P_1(\hat{w}_2, \hat{v}) = P_1(0, \hat{v})$. If $\|P_1(0, \hat{v})\|_{W_{-1}} \ge 4\beta_1 m$, then $|||P(\hat{w}_i, \hat{v})||| \ge |||P(0, \hat{v})||| \ge 2$ and therefore $d\big(|||P(\hat{w}_i, \hat{v})|||\big) = 0$. Consequently, we continue the above estimate by

$$\begin{aligned} &\|Q_1^{\hat{v}}\hat{w}_1 - Q_1^{\hat{v}}\hat{w}_2\|_{W_{-1}} \\ &\quad \le \Big| |||P(\hat{w}_1, \hat{v})||| - |||P(\hat{w}_2, \hat{v})||| \Big| 4\beta_1 m \\ &\quad = 4\beta_1 m \Big| \frac{1}{2m} \min\left(\frac{1}{\beta_2}, \frac{1}{4\beta_1\beta_2}\right) \big[\|P_2(\hat{w}_1, \hat{v})\|_{\widetilde{V}_{-1}} - \|P_2(\hat{w}_2, \hat{v})\|_{\widetilde{V}_{-1}}\big] \Big| \\ &\quad \le \frac{1}{2}\|\hat{w}_1 - \hat{w}_2\|_{W_1}, \end{aligned} \tag{2.15}$$

where we use the Lipschitz continuity of d with modulus 1 for the first inequality and the reverse triangle inequality for (2.15). From this we infer that $Q_1^{\hat{v}} : W_1 \to W_{-1}$ is Lipschitz continuous with constant $\frac{1}{2}$. Thus $M_{\hat{v}} : W_1 \to W_{-1}$ is also Lipschitz continuous and strongly monotone with constant $\frac{1}{2}$. Therefore, the mapping $S_1 : \widetilde{Y}_1 \to W_1$ given by

$$S_1(\hat{c}, \hat{v}) \;:=\; M_{\hat{v}}^{-1}\big[f^c - \tfrac{2}{\tau}(\hat{c} - c_a) - \nabla c_b \cdot F\overline{r}\big].$$

is well-defined.

3. Now we show that S_1 is totally continuous. For this purpose assume that $(\hat{c}_n, \hat{v}_n) \rightharpoonup (\hat{c}, \hat{v})$ in $\widetilde{Y}_1$ as $n \to +\infty$. Then $f_n := f^c - \frac{2}{\tau}(\hat{c}_n - c_a) - \nabla c_b \cdot F\overline{r}$ converges strongly to $f := f^c - \frac{2}{\tau}(\hat{c} - c_a) - \nabla c_b \cdot F\overline{r}$ in W_{-1} due to the compact embedding of

W_1 into W_{-1}. With $\hat{w}_n := S_1(\hat{c}_n, \hat{v}_n) = M_{\hat{v}}^{-1} f_n$, which implies $-\Delta \hat{w}_n + Q_1^{\hat{v}_n} \hat{w}_n = f_n$, and

$$\begin{aligned}\|Q_1^{\hat{v}_n} \hat{w}_n\|_{W_{-1}} &= \|d\big(|||P(\hat{w}_n, \hat{v}_n)|||\big) P_1(\hat{w}_n, \hat{v}_n)\|_{W_{-1}} \ \le\ 1 \cdot \|P_1(\hat{w}_n, \hat{v}_n)\|_{W_{-1}} \\ &\le \beta_1 \|\hat{v}_n\|_{\widetilde{V}_1}\end{aligned}$$

we obtain that $(Q_1^{\hat{v}_n} \hat{w}_n)$ and $(-\Delta \hat{w}_n)$ are bounded in W_{-1} and $(\hat{w}_n)$ is bounded in W_1. For any weakly converging subsequence $(\hat{w}_m)$ of $(\hat{w}_n)$ with weak limit $\hat{w} \in W_1$ it holds that $Q_1^{\hat{v}_m} \hat{w}_m \to Q_1^{\hat{v}} \hat{w}$ since P is totally continuous by Lemma 2.11. Consequently,

$$\begin{aligned}\|\hat{w}_m - \hat{w}\|_{W_1}^2 &\le \langle -\Delta(\hat{w}_m - \hat{w}), \hat{w}_m - \hat{w}\rangle_{W_1} \\ &= \langle f_m - f, \hat{w}_m - \hat{w}\rangle_{W_1} - \Big\langle Q_1^{\hat{v}_m} \hat{w}_m - Q_1^{\hat{v}} \hat{w}, \hat{w}_m - \hat{w}\Big\rangle_{W_1} \\ &\to 0 \quad \text{for } m \to \infty.\end{aligned}$$

Since $\hat{w}$ is the unique solution to $-\Delta \hat{w} + Q_1^{\hat{v}} \hat{w} = f$, S_1 must be totally continuous.

4. Consider the operator $R = (R_1, R_2) : \widetilde{Y}_1 \to \widetilde{Y}_{-1}$ given by

$$\begin{aligned}R_1(\hat{c}, \hat{v}) &:= (-\Delta)^{-1} Q_1(S_1(\hat{c}, \hat{v}), \hat{v}), \\ R_2(\hat{c}, \hat{v}) &:= Q_2(S_1(\hat{c}, \hat{v}), \hat{v}).\end{aligned}$$

We show now that R satisfies the assumptions of Proposition 2.8. We start by establishing the boundedness of R and (2.11). We use $d\big(|||P(\hat{w}, \hat{v})|||\big) = 0$ if $|||P(\hat{w}, \hat{v})||| \ge 2$ and $|||P(\hat{w}, \hat{v})||| \ge \gamma \|P(\hat{w}, \hat{v})\|_{\widetilde{Y}_{-1}}$ for $\gamma := \frac{1}{2m} \min(\frac{1}{\beta_1}, \min(\frac{1}{\beta_2}, \frac{1}{4\beta_1\beta_2}))$ to estimate

$$\|Q(\hat{w}, \hat{v})\|_{\widetilde{Y}_{-1}} \ = \ \big|d\big(|||P(\hat{w}, \hat{v})|||\big)\big| \, \|P(\hat{w}, \hat{v})\|_{\widetilde{Y}_{-1}} \le \frac{2}{\gamma}.$$

Consequently, R is bounded and satisfies (2.11). In order to realize that R is pseudomonotone it suffices to note that with S_1 and P also Q and R are totally continuous which implies that R is indeed pseudomonotone.

5. Note that A is maximal monotone by assumption. Therefore, by Lemma 2.10 and by Proposition 2.8 there exists a pair $(c, v) \in Y_1^2$ satisfying

$$\begin{gathered}\operatorname{Tr} v = r, \ c_0 = c_a, \ v_0 = v_a, \\ (-\Delta)^{-1} D^+ c + (-\Delta - I)\overline{c} + AS_+ c + R_1(\overline{c}, \overline{\overline{v}}) \ni (-\Delta)^{-1}(f^c - \nabla c_b \cdot F\overline{r}), \\ D^+ v - \Delta \overline{v} + (v_b \cdot \nabla)\overline{v} + R_2(\overline{c}, \overline{\overline{v}}) = f^v.\end{gathered}$$

Let us define $w := S_1(\overline{c}, \overline{\overline{v}})$. Thus we obtain

$$D^+ c - \Delta w + Q_1(w, \overline{\overline{v}}) + \nabla c_b \cdot F\overline{r} \ = \ \frac{2}{\tau}(\overline{c} - c_a) + M_{\overline{\overline{v}}} w + \nabla c_b \cdot F\overline{r} \ = \ f^c$$

which furthermore shows that

$$\begin{aligned}w = (-\Delta)^{-1}(-\Delta w) \ &= \ (-\Delta)^{-1}(f^c - \nabla c_b \cdot F\overline{r}) - (-\Delta)^{-1} D^+ c - R_1(\overline{c}, \overline{\overline{v}}) \\ &\in (-\Delta - I)\overline{c} + AS_+ c.\end{aligned}$$

Therefore, by (2.9), (2.10) and Corollary 2.7 we infer

$$\|w\|_{W_1}^2 + \|\overline{\overline{v}}\|_{\widetilde{V}_1}^2 \ \le \ m^2.$$

This implies

$$|||P(w,\overline{\overline{v}})||| \leq \frac{1}{2m}(\frac{1}{\beta_1}\beta_1\|\overline{\overline{v}}\|_{\widetilde{V}_1} + \frac{1}{\beta_2}\beta_2\|w\|_{W_1}) \leq \frac{1}{2m}\sqrt{2(\|w\|^2_{W_1} + \|\overline{\overline{v}}\|^2_{\widetilde{V}_1})} \leq 1.$$

Hence, from the definition of Q we conclude $d(|||P(w,\overline{\overline{v}})|||) = 1$ and thus $Q(w,\overline{\overline{v}}) = P(w,\overline{\overline{v}})$. This proves the assertion. □

THEOREM 2.13 (Existence of solutions of the semidiscrete CH-NS system). *For every* $(M,\tau,(c_a,v_a),(f^c,f^v),r) \in \mathbb{N}\times\mathbb{R}_+\times Y_1\times\widetilde{Y}^M_{-1}\times S^{M+1}_1$ *with* $\operatorname{Tr} v_a = r_0$ *and* $0<\tau\leq\tau_0(\Omega)$ *and for every proper, convex and lower-semicontinuous functional* $\varphi : W_1 \to \overline{\mathbb{R}}$ *satisfying* (H$_1$) *for a constant* $C_\varphi \in \mathbb{R}$*, the set* $\mathcal{S}(M,\tau,(c_a,v_a),(f^c,f^v),r,\partial\varphi)$ *is non-empty.*

Proof. We prove this theorem by induction over M. For $M=0$, the assertion is immediate. Hence, we consider $(M+1,\tau,(c_a,v_a),(f^c,f^v),r) \in \mathbb{N}\times\mathbb{R}_+\times Y_1\times Y^{M+1}_{-1}\times S^{M+2}_1$ with $\operatorname{Tr} v_a = r_0$ and $0<\tau\leq\tau_0(\Omega)$.
Assuming that we have $(c^1,v^1)\in\mathcal{S}(M,\tau,(c_a,v_a),(S_-f^c,S_-f^v),S_-r,\partial\varphi)$, by Proposition 2.12, there exists $(c^2,v^2) \in \mathcal{S}(1,\tau,(c^1_M,v^1_M),(f^c_{M+1},f^v_{M+1}),(r_M,r_{M+1}),\partial\varphi)$. Therefore $(c,v) := ((c^1_0,...,c^1_M,c^2_1),(v^1_0,...,v^1_M,v^2_1))$ is an element of the solution set $\mathcal{S}(M+1,\tau,(c_a,v_a),(f^c,f^v),r,\partial\varphi)$. □

2.3. Energy estimates II. In order pass to the limit in the semidiscrete CH-NS system with approximating sequences, we need some a priori estimates for the energy. These are proven next.

PROPOSITION 2.14. *Consider* $M\in\mathbb{N}$*,* $\tau>0$*, the initial data* $(c_a,v_a)\in Y_1$ *as well as bounds* $C_r, C_\varphi, C_{c_a}\in\mathbb{R}$*. Then there exists a constant* C *depending only on* $\Omega,\tau,M,(c_a,v_a),C_r,C_\varphi$ *and* C_{c_a} *such that for all* $r\in S^{M+1}_1$ *with* $\|Fr\|_{V_1^{M+1}}\leq C_r$*, and all convex functionals* $\varphi : W_1\to\overline{\mathbb{R}}$ *satisfying* (H$_1$) *with constant* C_φ *and* $\varphi(c_a)\leq C_{c_a}$*, every solution* $(c,w,v)\in\mathcal{S}^w(M,\tau,(c_a,v_a),0,r,\partial\varphi)$ *is bounded such that* $\|(c,w,v)\|_{(W_1\times W_1\times V_1)^{M+1}}\leq C$ *and* $\max\{\mathrm{E}(c_i,v_i) : i=0,...,M\}\leq C$*.*

Proof. We prove the proposition by induction over M. For $M=0$, the assertion is obvious. Now, suppose it is valid for some $M\in\mathbb{N}$. Let a solution $(c,w,v)\in\mathcal{S}^w(M+1,\tau,(c_a,v_a),0,r,\partial\varphi)$ be given with $r\in S^{M+2}_1$ and $\|Fr\|_{V_1^{M+2}}\leq C_r$, and with $\varphi : W_1\to\overline{\mathbb{R}}$ convex such that $\varphi(f)-\frac{1}{2}\|f\|^2_{W_0}\geq C_\varphi$ and $\varphi(c_a)\leq C_{c_a}$. By our induction hypothesis there exists a constant C_1 depending only on $\Omega,\tau,M,(c_a,v_a),C_r,C_\varphi$ and C_{c_a} such that $\|(c,w,v)\|_{X^{M+1}}\leq C_1$ and $\max\{\mathrm{E}(c_i,v_i) : i=0,...,M\}\leq C_1$. Here, we use the definition $X := W_1\times W_1\times V_1$ and the notation $\|(c,w,v)\|_{X^{M+1}} = \|((c_0,...,c_M),(w_0,...,w_M),(v_0,...,v_M))\|_{X^M}$.
By Lemma 2.6 there exists a constant C_2 depending only on Ω,τ and C_r with

$$\begin{aligned}\mathrm{E}(c_{M+1},v_{M+1}) &+ \tfrac{\tau}{2}(\|w_{M+1}\|^2_{W_1} + \|\overline{\overline{v}}_M\|^2_{\widetilde{V}_1})\\ &\leq \mathrm{E}(c_M,v_M) + C_2(\|(c_M,v_M)\|^2_{Y_1}+1)\\ &\leq C_1 + C_1^2C_2 + C_2 \;=:\; C_3,\end{aligned}$$

where we use that $\|BF\overline{r}_M\|_{\widetilde{V}_{-1}} = \|(v_M\cdot\nabla)F\overline{r}_M\|_{\widetilde{V}_{-1}}\leq C\|v_M\|_{V_1}\|F\overline{r}_M\|_{V_1}$. In particular, $\mathrm{E}(c_{M+1},v_{M+1})\leq C_3$. Because of $\mathrm{E}(c_{M+1},v_{M+1})\geq\mathrm{E}_{\text{free}}(c_{M+1})\geq\frac{1}{2}\|c_{M+1}\|^2_{W_1}+C_\varphi\geq C_\varphi$, we have

$$\|c_{M+1}\|^2_{W_1} \;\leq\; 2(C_3 - C_\varphi)$$

as well as

$$\|w_{M+1}\|^2_{W_1} + \|\overline{\overline{v}}_M\|^2_{\widetilde{V}_1} \;\leq\; \tfrac{2}{\tau}(C_3 - \mathrm{E}(c_{M+1},v_{M+1})) \;\leq\; \tfrac{2}{\tau}(C_3 - C_\varphi).$$

Finally recalling $C_P := \|\operatorname{Id}\|_{\mathcal{L}(\widetilde{V}_1;V_1)}$, it follows that

$$\begin{aligned}\|v_{M+1}\|_{V_1}^2 &= \|\widetilde{v}_{M+1} + Fr_{M+1}\|_{V_1}^2 \;=\; \|2\tfrac{1}{2}(\widetilde{v}_M + \widetilde{v}_{M+1}) - \widetilde{v}_M + Fr_{M+1}\|_{V_1}^2 \\ &\le 2\big(4\|\overline{\widetilde{v}}_M\|_{V_1}^2 + \|v_M - F(r_M + r_{M+1})\|_{V_1}^2\big) \\ &\le 8C_P\|\overline{\widetilde{v}}_M\|_{\widetilde{V}_1}^2 + 4(\|v_M\|_{V_1}^2 + \|F(r_M + r_{M+1})\|_{V_1}^2) \\ &\le \tfrac{16}{\tau}C_P(C_3 - C_\varphi) + 4(C_1^2 + 4C_r^2).\end{aligned}$$

This completes the proof. □

3. Optimal control of the semidiscrete CH-NS system. Now we are ready to state the optimization problem under investigation, prove existence of a solution and establish a suitable stationary characterization. For the application of the theory developed in section 2 we invoke the following assumption.

ASSUMPTION 3.1. *The space U_1 is a closed, linear subspace of S_1 with induced inner product, and with its dual denoted by $U_{-1} := U_1^*$. We fix $M \in \mathbb{N}$, initial values $(c_a, v_a) \in Y_1$ with $c_a \in H^2(\Omega)$ and $Tr\, v_a \in U_1$, right hand sides $(f^c, f^v) \in Y_{-1}$, $\gamma \ge 0$ and a desired concentration $c_e \in W_1$. Moreover, let $\tau_0 > 0$ be given as in Lemma 2.10 and τ such that $0 < \tau \le \tau_0$.*

PROBLEM 3.2. *Given a proper, convex and lower-semicontinuous functional $\varphi : W_1 \to \overline{\mathbb{R}}$ we consider the problem* (P_φ)

$$(\mathrm{P}_\varphi) \qquad \inf\{J(c,v,u) \mid (c,v,u) \in W_1^{M+1} \times V_1^{M+1} \times U_1^{M+1},\ (c,v) \in \mathcal{S}_\varphi u\},$$

where the functional $J : W_1^{M+1} \times V_1^{M+1} \times U_1^{M+1} \to \mathbb{R}$ and the set $\mathcal{S}_\varphi u$ for $u \in U_1^{M+1}$ are given by

$$\begin{aligned}J(c,v,u) &:= \frac{1}{2}\Big(\gamma\|c_M - c_e\|_{W_0}^2 + \|u\|_{U_1^{M+1}}^2\Big), \\ \mathcal{S}_\varphi u &:= \mathcal{S}(M, \tau, (c_a, v_a), (f^c, f^v), u, \partial\varphi).\end{aligned}$$

Moreover, we write $\mathcal{S}_\varphi^w u := \mathcal{S}^w(M, \tau, (c_a, v_a), (f^c, f^v), u, \partial\varphi)$.

Before we address the existence of minimizers of problem (P_φ) we need a closedness result for the solution sets along a sequence $(u^{(n)})$ and for a sequence $(\varphi^{(n)})$ approximating $\varphi^{(\infty)} := \varphi$.

PROPOSITION 3.1. *Assume we are given a bounded sequence $(u^{(n)})_{n\in\mathbb{N}}$ in U_1^{M+1} and a sequence $(\varphi^{(n)})_{n\in\mathbb{N}^*}$ of proper, convex and lower-semicontinuous functionals from W_1 into $\overline{\mathbb{R}}$ satisfying* (H_1) *of Corollary 2.7 for a common constant $C = C_{\varphi^{(n)}}$ for all $n \in \mathbb{N}$. In addition, suppose that sequence $(A^{(n)})$ with $A^{(n)} := \partial\varphi^{(n)} \subset W_1 \times W_{-1}$ for $n \in \mathbb{N}^*$ fulfills one of the follwing conditions:*

(H_2) *Whenever $A^{(m)} \ni (\hat{c}^{(m)}, \hat{a}^{(m)}) \rightharpoonup (\hat{c}^{(\infty)}, \hat{a}^{(\infty)})$ in $W_1 \times W_{-1}$ for a subsequence $m \in M \subset \mathbb{N}$ with*

$$\varlimsup_{M \ni m \to \infty} \Big\langle \hat{a}^{(m)} - \hat{a}^{(\infty)}, \hat{c}^{(m)} - \hat{c}^{(\infty)} \Big\rangle_{W_1} \;\le\; 0,$$

then $(\hat{c}^{(\infty)}, \hat{a}^{(\infty)}) \in A^{(\infty)}$ and $\langle \hat{a}^{(m)} - \hat{a}^{(\infty)}, \hat{c}^{(m)} - \hat{c}^{(\infty)} \rangle_{W_1} \to 0$.

(H_3) *It holds that $A^{(n)}(W_1) \subset W_0$ and $\big(-\Delta\hat{c}^{(n)} | \hat{a}^{(n)}\big)_{W_0} \ge 0$ for all $n \in \mathbb{N}$ and all $(\hat{c}^{(n)}, \hat{a}^{(n)}) \in A^{(n)}$ with $-\Delta\hat{c}^{(n)} \in W_0$. Moreover, whenever $A^{(m)} \ni (\hat{c}^{(m)}, \hat{a}^{(m)}) \rightharpoonup (\hat{c}^{(\infty)}, \hat{a}^{(\infty)})$ in $W_1 \times W_0$ for a subsequence $m \in M \subset \mathbb{N}$, then $(\hat{c}^{(\infty)}, \hat{a}^{(\infty)}) \in A^{(\infty)}$.*

Then there exists a subsequence $(u^{(m)})$ *of* $(u^{(n)})$ *converging weakly to* $u^{(\infty)}$ *in* U_1^{M+1} *and a sequence of solutions* $(c^{(n)}, w^{(n)}, v^{(n)}) \in \mathcal{S}^w_{\varphi^{(n)}} u^{(n)}$ *for* $n \in \mathbb{N}^*$ *such that* $(c^{(m)})$ *converges strongly in* W_1^{M+1} *to* $c^{(\infty)}$ *and* $(w^{(m)}, v^{(m)})$ *converges weakly in* $(W_1 \times V_1)^{M+1}$ *to* $(w^{(\infty)}, v^{(\infty)})$*. Moreover, if* (H$_3$) *holds true, then the sequences* $(c^{(n)})$ *and* $(S_+ a^{(n)})$ *are bounded in* $H^2(\Omega)^{M+1}$ *respectively* W_0^M *for* $a^{(n)} \in A^{(n)} c^{(n)}$ *given by*

$$S_+ a^{(n)} = (-\Delta)^{-1} f^c - [(-\Delta)^{-1} D^+ c^{(n)} + (-\Delta - I)\overline{c}^{(n)} + (-\Delta)^{-1} \nabla S_- c^{(n)} \cdot \overline{v}^{(n)}].$$

Proof. By Theorem 2.13 we can find a solution $(c^{(n)}, w^{(n)}, v^{(n)}) \in \mathcal{S}^w_{\varphi^{(n)}} u^{(n)}$ for every $n \in \mathbb{N}$ satisfying

$$\begin{aligned} \operatorname{Tr} v^{(n)} = u^{(n)},\ \ c_0^{(n)} = c_a,\ \ w_0^{(n)} = 0,\ \ v_0^{(n)} = v_a, \\ (-\Delta)^{-1} D^+ c^{(n)} + (-\Delta - I)\overline{c}^{(n)} + S_+ a^{(n)} + (-\Delta)^{-1} \nabla S_- c^{(n)} \cdot \overline{v}^{(n)} \ =\ (-\Delta)^{-1} f^c, \\ S_+ w^{(n)} \ =\ (-\Delta)^{-1} \big[f^c - (D^+ c^{(n)} + \nabla S_- c^{(n)} \cdot \overline{v}^{(n)}) \big], \\ D^+ v^{(n)} - \Delta \overline{v}^{(n)} + (S_- v^{(n)} \cdot \nabla) \overline{v}^{(n)} - S_+ w^{(n)} \nabla S_- c^{(n)} \ =\ f^v \end{aligned}$$

for $a^{(n)} \in A^{(n)} c^{(n)}$. Proposition 2.14 shows that the sequence $(c^{(n)}, w^{(n)}, v^{(n)})$ is bounded in $(W_1 \times W_1 \times V_1)^{M+1}$. Hence, we can pass to a subsequence $(c^{(m)}, w^{(m)}, v^{(m)}, u^{(m)})$ that converges weakly to some $(c^{(\infty)}, w^{(\infty)}, v^{(\infty)}, u^{(\infty)})$ in $(W_1 \times W_1 \times V_1 \times U_1)^{M+1}$. Exploiting the weak continuity of linear, bounded operators and the total continuity of $(\hat{c}, \hat{v}) \mapsto \nabla \hat{c} \cdot \hat{v}$, $(\hat{c}, \hat{v}) \mapsto \hat{w} \nabla \hat{c}$ and $(\hat{v}_1, \hat{v}) \mapsto (\hat{v}_1 \cdot \nabla)\hat{v}$ we conclude that $S_+ a^{(m)}$ converges weakly to some $S_+ a^{(\infty)}$ in W_{-1}^M and

$$\begin{aligned} \operatorname{Tr} v^{(\infty)} = u^{(\infty)},\ \ c_0^{(\infty)} = c_a,\ \ w_0^{(\infty)} = 0,\ \ v_0^{(\infty)} = v_a, \\ (-\Delta)^{-1} D^+ c^{(\infty)} + (-\Delta - I)\overline{c}^{(\infty)} + S_+ a^{(\infty)} + (-\Delta)^{-1} \nabla S_- c^{(\infty)} \cdot \overline{v}^{(\infty)} \ =\ (-\Delta)^{-1} f^c, \\ S_+ w^{(\infty)} \ =\ (-\Delta)^{-1} \big[f^c - (D^+ c^{(\infty)} + \nabla S_- c^{(\infty)} \cdot \overline{v}^{(\infty)}) \big], \\ D^+ v^{(\infty)} - \Delta \overline{v}^{(\infty)} + (S_- v^{(\infty)} \cdot \nabla) \overline{v}^{(\infty)} - S_+ w^{(\infty)} \nabla S_- c^{(\infty)} \ =\ f^v. \end{aligned}$$

In order to finish the proof, it remains to show the boundedness of $(c^{(n)})$ and $(a^{(n)})$ in $H^2(\Omega)$, respectively W_0, in case of (H$_3$), the strong convergence properties and that $(c_i^{(\infty)}, a_i^{(\infty)}) \in A^{(\infty)}$ for $i = 1, ..., M$. This will be done by induction over i together with $-\Delta c_i^{(m)} \in W_0$ for the case of (H$_3$). We obviously have $c_0^{(n)} = c_a \in V_1 \cap H^2(\Omega)$ for all $n \in \mathbb{N}^*$. This concludes the basis step of the induction. Let us suppose that $c_i^{(m)} \to c_i^{(\infty)}$ in W_1 for some $i = 0, ..., M-1$ as well as $w_i^{(m)} \to w_i^{(\infty)}$ in W_1 and $v_i^{(m)} \to v_i^{(\infty)}$ in V_1. Using the equations above we obtain

$$\begin{aligned} & -\Delta(\overline{c}_i^{(m)} - \overline{c}_i^{(\infty)}) + a_{i+1}^{(m)} - a_{i+1}^{(\infty)} \\ & \quad = -\Big[(-\Delta)^{-1} (D^+ c_i^{(m)} - D^+ c_i^{(\infty)}) - I(\overline{c}_i^{(m)} - \overline{c}_i^{(\infty)}) \\ & \qquad + (-\Delta)^{-1} (\nabla S_- c_i^{(m)} \cdot \overline{c}_i^{(m)} - \nabla S_- c_i^{(\infty)} \cdot \overline{c}_i^{(\infty)}) \Big] \end{aligned}$$

and therefore by the compactness of $(-\Delta)^{-1}, I : W_1 \to W_{-1}$ and $(\hat{v}_1, \hat{v}) \mapsto (\hat{v}_1 \cdot \nabla)\hat{v} : V_1 \times V_1 \to \widetilde{V}_{-1}$ that

$$\begin{aligned} \lim_{m \to \infty} \Big[\frac{1}{2} \Big(\Big\langle -\Delta(c_{i+1}^{(m)} - c_{i+1}^{(\infty)}), c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \Big\rangle_{W_1} + \Big\langle -\Delta(c_i^{(m)} - c_i^{(\infty)}), c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \Big\rangle_{W_1} \Big) \\ + \Big\langle a_{i+1}^{(m)} - a_{i+1}^{(\infty)}, c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \Big\rangle_{W_1} \Big] \ =\ 0. \end{aligned}$$

Since $c_i^{(m)} \to c_i^{(\infty)}$ in W_1 we find

$$\begin{aligned}
\overline{\lim_{m\to\infty}} &\left\langle a_{i+1}^{(m)} - a_{i+1}^{(\infty)}, c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \right\rangle_{W_1} \\
&\le -\underline{\lim}_{m\to\infty} \left\langle -\Delta(c_{i+1}^{(m)} - c_{i+1}^{(\infty)}), c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \right\rangle_{W_1} \\
&= -\underline{\lim}_{m\to\infty} \|c_{i+1}^{(m)} - c_{i+1}^{(\infty)}\|_{W_1}^2 \le 0.
\end{aligned} \tag{3.1}$$

If (H$_2$) is satisfied, if follows directly that $(c_{i+1}^{(\infty)}, a_{i+1}^{(\infty)}) \in A^{(\infty)}$ as well as the convergence $\left\langle a_{i+1}^{(m)} - a_{i+1}^{(\infty)}, c_{i+1}^{(m)} - c_{i+1}^{(\infty)} \right\rangle_{W_1} \to 0$ and therefore $c_{i+1}^{(n)} \to c_{i+1}^{(\infty)}$ in W_1. In the case of (H$_3$), notice that from

$$\begin{aligned}
-\frac{1}{2}\Delta c_{i+1}^{(n)} + a_{i+1}^{(n)} &= -\Delta \overline{c}_i^{(n)} + a_{i+1}^{(n)} + \frac{1}{2}\Delta c_i^{(n)} \\
&= (-\Delta)^{-1}(f_i^c - \nabla c_{i-1}^{(n)} \cdot \overline{v}_i^{(n)} - D^+ c_i^{(n)}) + I\overline{c}_i^{(n)} + \frac{1}{2}\Delta c_i^{(n)} \\
&= w_{i+1}^{(n)} + I\overline{c}_i^{(n)} + \frac{1}{2}\Delta c_i^{(n)} =: g^{(n)}
\end{aligned} \tag{3.2}$$

and $g^{(n)} \in W_0$ as well as $A^{(n)}(W_1) \subset W_0$ it follows that $-\Delta c_{i+1}^{(n)} \in W_0$. Moreover, since $(g^{(n)})$ is bounded in W_0, the assumption (H$_3$) yields

$$\begin{aligned}
\|a_{i+1}^{(n)}\|_{W_0}^2 = \left(g^{(n)} + \tfrac{1}{2}\Delta c_{i+1}^{(n)} | a_{i+1}^{(n)}\right)_{W_0} &\le \left(g^{(n)} | a_{i+1}^{(n)}\right)_{W_0} \\
&\le \|g^{(n)}\|_{W_0} \|a_{i+1}^{(n)}\|_{W_0}.
\end{aligned}$$

Consequently, $(a_{i+1}^{(n)})$ is bounded in W_0 and therefore $(c_{i+1}^{(n)})$ in $H^2(\Omega)$ by (3.2). Hence, we can assume without loss of generality that $(a_{i+1}^{(m)})$ converges weakly in W_0 and strongly in W_{-1} to $a_{i+1}^{(\infty)}$. By (H$_3$) it follows that $(c_{i+1}^{(\infty)}, a_{i+1}^{(\infty)}) \in A^{(\infty)}$ and by (3.1) that $c_{i+1}^{(m)} \to c_{i+1}^{(\infty)}$ in W_1. This completes the proof. ∎

Remark 3.3. *It is well known that for a maximal monotone operator $A \subset W_1 \times W_{-1}$ its Yosida approximations $A^{(n)} := A_{\lambda_n}$ with parameter $\lambda_n := \frac{1}{n}$ as well as the sequence $A^{(n)} := A$ itself satisfy the condition* (H$_2$). *This condition is used in order to show the existence of minimizers for a fixed potential φ, whereas* (H$_3$) *will be the appropriate condition for the approximation procedure which we apply below.*

The existence result is stated next.

Theorem 3.2. *For every proper, convex and lower-semicontinuous functional $\varphi : W_1 \to \overline{\mathbb{R}}$ fulfilling* (H$_1$), *problem* (P$_\varphi$) *admits a minimizer.*

Proof. Let $(c^{(n)}, u^{(n)}) \in U_1^{M+1} \times W_1^{M+1}$ be an infimizing sequence for problem (P$_\varphi$) and $(w^{(n)}, v^{(n)}) \in W_1^{M+1} \times V_1^{M+1}$ such that $(c^{(n)}, w^{(n)}, v^{(n)}) \in \mathcal{S}_\varphi u^{(n)}$. The coercivity of J in the second component and the boundedness imply that $(u^{(n)})$ is bounded in U_1^{M+1}. We choose $\varphi^{(n)} := \varphi$ in Proposition 3.1 and Remark 3.3 to conclude the existence of a weakly convergent subsequence $(c^{(n)}, w^{(n)}, v^{(n)})$ in $(W_1 \times W_1 \times V_1)^{M+1}$ with limit within $\mathcal{S}^w_{\varphi^{(n)}} u^{(\infty)}$ and $(u^{(n)})$ in U_1^{M+1} with limit $u^{(\infty)}$. The weak lower semicontinuity of J implies that this limit is indeed a minimizer of (P$_\varphi$). This finishes the proof. ∎

For the proof of the subsequent theorem we need the following auxiliary result.

Lemma 3.3. *Let $\varphi : W_1 \to \overline{\mathbb{R}}$ be a proper, convex and lower-semicontinuous functional with a single-valued subdifferential $A := \partial\varphi$, which is defined on all of*

W_1. Moreover, suppose that $A : W_1 \to W_{-1}$ is continuously Fréchet differentiable in $w_0 \in W_1$ with derivative $Q := DA(w_0) \in \mathcal{L}(W_1; W_{-1})$. Then $\varphi_0(w) := \frac{1}{2}\langle Qw, w\rangle_{W_1}$ is a proper, convex and lower-semicontinuous functional on W_1 and its subdifferential $A_0 := \partial\varphi_0$ is single-valued, defined on all W_1, and continuously Fréchet differentiable with $DA_0(w) = Q$ for all $w \in W_1$.

Proof. Since Aw is a singelton and continuous for every $w \in W_1$, φ is Gâteaux differentiable (cf. Showalter [31]). Consequently, $Q \in \mathcal{L}(W_1; W_{-1})$ is symmetric and positive. Hence, the assertion follows. ∎

Next we study the adjoint system pertinent to (P_φ). This system is relevant for deriving first order optimality conditions of approximate versions of (P_φ) with a smooth potential φ.

THEOREM 3.4. *Assume that (c^o, v^o, u^o) is a minimizer of (P_φ) and that $A = \partial\varphi \subset W_1 \times W_{-1}$ is single-valued, defined on W_1 and continuously Fréchet differentiable. Then there exists $(p, q) \in \widetilde{Y}_1^M$ such that*

$$\begin{aligned}
0 &= -(-\Delta)^{-1}D^+p + (-\Delta - I)\overline{p} + DA(S_+c^o)^*S_-p - \operatorname{div}((-\Delta)^{-1}(S_+p)S_+\overline{v}^o)\\
&\quad - \operatorname{div}((-\Delta)^{-1}D^+(S_+c^o)S_+q),\\
\gamma(c^o_M - c_e) &= \Big(\frac{1}{\tau}(-\Delta)^{-1} + \frac{1}{2}(-\Delta - I) + DA(c^o_M)^*\Big)p_{M-1},
\end{aligned}$$

$$\begin{aligned}
S_+\, Tr^* P^*_{U_1} u^o &= -I^*D^+q - \Delta^*\overline{q} + b_1(S_+\overline{v}^o, S_+q) + \frac{1}{2}\Big(b_2(S_-v^o, S_-q) + b_2(S_+v^o, S_+q)\Big)\\
&\quad + \frac{1}{2}\Big((-\Delta)^{-1}(\nabla S_-c^o \cdot S_-q)\nabla S_-c^o + (-\Delta)^{-1}(\nabla S_+c^o \cdot S_+q)\nabla S_+c^o\Big)\\
&\quad + \frac{1}{2}\Big((-\Delta)^{-1}(S_-p)\nabla S_-c^o + (-\Delta)^{-1}(S_+p)\nabla S_+c^o\Big),\\
Tr^* P^*_{U_1} u^o_M &= \frac{1}{\tau}I^*q_{M-1} + \frac{1}{2}\Big[-\Delta^*q_{M-1} + b_2(v^o_{M-1}, q_{M-1})\\
&\quad + (-\Delta)^{-1}(\nabla c^o_{M-1} \cdot q_{M-1})\nabla c^o_{M-1} + (-\Delta)^{-1}p_{M-1}\nabla c^o_{M-1}\Big],
\end{aligned}$$

where the last two equations are understood in the sense of $(Z^*)^{M-1}$ and Z^* for $Z := \{v \in V_1 : P_{U_1^\perp}\, Tr\, v = 0\}$ with P_{U_1} and $P_{U_1^\perp}$ denoting the orthogonal projections of S_1 onto U_1 and its orthogonal complement $U_1^\perp$.

Proof. We split the proof into three steps. Our goal is to apply the theory developed in Zowe/Kurcyusz [36]. For this purpose, in the first step of the proof we define relevant quantities. Then we apply [36] in the second step. And finally, in the third step, we rearrange terms in order to derive the asserted adjoint system.

1. In the proof we utilize following sets and spaces

$$\begin{aligned}
X_1 &:= W_1^{M+1} \times V_1^{M+1} \times U_1^{M+1},\\
X_2 &:= W_{-1}^M \times \widetilde{V}_{-1}^M \times U_{-1}^M,\\
C_{(c_b, v_b)} &:= \{(c, v, u) \in X_1 \;:\; (c_0, v_0) = (c_b, v_b),\;\; P_{U_1^\perp}\operatorname{Tr} v = 0\},
\end{aligned}$$

and the mappings $Q : X_1 \to X_2$ and $g : X_1 \to \mathbb{R}$ defined by

$$\begin{aligned}
Q(c, v, u) &:= \Big((-\Delta)^{-1}D^+c + (-\Delta - I)\overline{c} + AS_+c + (-\Delta)^{-1}\nabla S_-c \cdot \overline{v} - (-\Delta)^{-1}f^c\,,\\
&\qquad D^+v - \Delta\overline{v} + (S_-v \cdot \nabla)\overline{v} - (-\Delta)^{-1}(f^v - D^+c - \nabla S_-c \cdot \overline{v})\nabla S_-c - f^v\,,\\
&\qquad S_+J_{U_1}(P_{U_1}\operatorname{Tr} v - u)\Big),\\
g(c, v, u) &:= J(c, v, u),
\end{aligned}$$

where J_{U_1} denotes the duality mapping from U_1 to U_{-1}. Their derivatives at $(c, v, u) \in X_1$ in direction $(c^\delta, v^\delta, u^\delta) \in X_1$ read as follows

$$
\begin{aligned}
&DQ(c, v, u; c^\delta, v^\delta, u^\delta) \\
&= \Big((-\Delta)^{-1} D^+ c^\delta + (-\Delta - I)\overline{c}^\delta + DA(S_+ c; S_+ c^\delta) + (-\Delta)^{-1}[\nabla S_- c^\delta \cdot \overline{v} + \nabla S_- c \cdot \overline{v}^\delta], \\
&\quad D^+ v^\delta - \Delta \overline{v}^\delta + (S_- v^\delta \cdot \nabla)\overline{v} + (S_- v \cdot \nabla)\overline{v}^\delta + (-\Delta)^{-1}(D^+ c + \nabla S_- c \cdot \overline{v})\nabla S_- c^\delta \\
&\qquad + (-\Delta)^{-1}(D^+ c^\delta + \nabla S_- c^\delta \cdot \overline{v} + \nabla S_- c \cdot \overline{v}^\delta)\nabla S_- c, \\
&\quad S_+ J_{U_1}(P_{U_1} \operatorname{Tr} v^\delta - u^\delta) \Big), \\
&Dg(c, v, u; c^\delta, v^\delta, u^\delta) \;=\; \gamma \left(c_M - c_e | c^\delta_M\right)_{W_0} + \left(u | u^\delta\right)_{U_1^{M+1}}.
\end{aligned}
$$

2. The triple (c^o, v^o, u^o) is a minimizer of g on the set $Z := C_{(c_a, v_a)} \cap Q^{-1}(0)$. For the application of the existence result of Lagrange multipliers by Zowe/Kurcyusz [36], it has to be shown that (c^o, v^o, u^o) is regular in the sense of [36]. For this purpose we fix an arbitrary $(\hat{f}^c, \hat{f}^v, \hat{f}^u) \in X_2$ and show the existence of a $(c^\delta, v^\delta, u^\delta) \in C_{(0,0)}$ such that $DQ(c^o, v^o, u^o; c^\delta, v^\delta, u^\delta) = (\hat{f}^c, \hat{f}^v, \hat{f}^u)$. The latter equation is equivalent to

$$c^\delta_0 = 0, \qquad v^\delta_0 = 0, \qquad J_{U_1}(P_{U_1}(\operatorname{Tr} v^\delta - u^\delta)) = \hat{f}^u, \tag{3.3}$$
$$(-\Delta)^{-1} D^+ c^\delta + (-\Delta - I)\overline{c}^\delta + DA(S_+ c^o; S_+ c^\delta) + (-\Delta)^{-1} \nabla S_- c^o \cdot \overline{v}^\delta \;=\; g^c(c^\delta, v^\delta, u^\delta), \tag{3.4}$$
$$D^+ v^\delta - \Delta \overline{v}^\delta + (S_- v^o \cdot \nabla)\overline{v}^\delta + (-\Delta)^{-1}(D^+ c^\delta + \nabla S_- c^o \cdot \overline{v}^\delta)\nabla S_- c^o \;=\; g^v(c^\delta, v^\delta, u^\delta), \tag{3.5}$$

where the right hand sides g^c and g^v correspond to

$$
\begin{aligned}
g^c(c^\delta, v^\delta, u^\delta) &:= \hat{f}^c - (-\Delta)^{-1} \nabla S_- c^\delta \cdot \overline{v}^o, \\
g^v(c^\delta, v^\delta, u^\delta) &:= \hat{f}^v - \Big[(S_- v^\delta \cdot \nabla)\overline{v}^o + (-\Delta)^{-1}(D^+ c + \nabla S_- c^o \cdot \overline{v}^o)\nabla S_- c^\delta \\
&\qquad\qquad + (-\Delta)^{-1} \nabla S_- c^\delta \cdot \overline{v}^o \nabla S_- c^o \Big].
\end{aligned}
$$

As in Theorem 2.13, the existence of a triple $(c^\delta, v^\delta, u^\delta)$ satisfying the above equations will be proved by induction over M'. Moreover, we show that it is possible to satisfy the additional condition $\operatorname{Tr} v^\delta = 0$. In case of $M' = 0$, the conclusion is evident. Now, we assume that the system is satisfied for some M' with $0 \le M' < M$. We apply the previous lemma and Proposition 2.12 in order to conclude the existence of $(c^2, v^2) \in (W_1 \times V_1)^2$ such that

$$
\begin{aligned}
&c^2_0 = c_{M'}, \qquad v^2_0 = v_{M'}, \qquad \operatorname{Tr} v^2 = 0, \qquad u^2 = -J^{-1}_{U_1}(\hat{f}^u_{M'}, \hat{f}^u_{M'+1}), \\
&(-\Delta)^{-1} D^+ c^2 + (-\Delta - I)\overline{c}^2 + DA(S_+ c^o; S_+ c^2) + (-\Delta)^{-1} \nabla S_- c^o \cdot \overline{v}^2 \;=\; g^c(c^\delta, v^\delta, u^\delta), \\
&D^+ v^2 - \Delta \overline{v}^2 + (S_- v^o \cdot \nabla)\overline{v}^2 + (-\Delta)^{-1}(D^+ c^2 + \nabla S_- c^o \cdot \overline{v}^2)\nabla S_- c^o \;=\; g^v(c^\delta, v^\delta, u^\delta).
\end{aligned}
$$

Consequently, $((c_0, ..., c_{M'}, c^2_1), (v_0, ..., v_{M'}, v^2_1), (u_0, ..., u_{M'}, u^2_1))$ solves the system (3.3)–(3.5) for $M' + 1$.

3. The result of Zowe/Kurcyusz now implies that for some $(p, q, r) \in X_2^* \cong W_1^M \times \widetilde{V}_1^M \times U_1^M$

$$Dg(c^o, v^o, u^o; c^\delta, v^\delta, u^\delta) \;=\; \left\langle DQ(c^o, v^o, u^o; c^\delta, v^\delta, u^\delta), (p, q, r) \right\rangle_{X_2^*}$$

for all $(c^\delta, v^\delta, u^\delta) \in C_{(0,0)}$ holds true.

First notice that for $c_1, c_2 \in W_1$, $v \in V_1$ and $\widetilde{v} \in \widetilde{V}_1$ we have

$$\begin{aligned}\big\langle(-\Delta)^{-1}(\nabla c_1\cdot v), c_2\big\rangle_{W_1} &= \big\langle c_2, (-\Delta)^{-1}(\nabla c_1\cdot v)\big\rangle_{W_1} = \big\langle \nabla c_1\cdot v, (-\Delta)^{-1}c_2\big\rangle_{W_1}\\ &= \big\langle -\operatorname{div}((-\Delta)^{-1}c_2 v), c_1\big\rangle_{W_1} = \big\langle(-\Delta)^{-1}c_2\nabla c_1, v\big\rangle_{V_1}\end{aligned}$$

as well as

$$\begin{aligned}&\big\langle(-\Delta)^{-1}(\nabla c_2\cdot v)\nabla c_1, \widetilde{v}\big\rangle_{\widetilde{V}_1}\\ &\quad= \big\langle \nabla c_1\cdot\widetilde{v}, (-\Delta)^{-1}(\nabla c_2\cdot v)\big\rangle_{W_1} = \big\langle\nabla c_2\cdot v, (-\Delta)^{-1}(\nabla c_1\cdot\widetilde{v})\big\rangle_{W_1}\\ &\quad= \big\langle(-\Delta)^{-1}(\nabla c_1\cdot\widetilde{v})\nabla c_2, v\big\rangle_{V_1}.\end{aligned}$$

Choosing $(c^\delta, 0, 0) \in C_{(0,0)}$, passing to adjoint operators and collecting terms involving c_i^δ we obtain

$$\begin{aligned}&\gamma\left(c_M^o - c_e | c_M^\delta\right)_{W_0}\\ &= \big\langle(-\Delta)^{-1}D^+c^\delta + (-\Delta - I)\overline{c}^\delta + DA(S_+c^o; S_+c^\delta) + (-\Delta)^{-1}(\nabla S_-c^\delta\cdot\overline{v}^o), p\big\rangle_{W_1^M}\\ &\quad+ \big\langle(-\Delta)^{-1}[D^+c^o]\nabla S_-c^\delta, q\big\rangle_{\widetilde{V}_1^{M+1}}\\ &= \sum_{i=1}^{M-1}\Big\langle \frac{1}{\tau}(-\Delta)^{-1}(p_{i-1}-p_i) + \frac{1}{2}(-\Delta - I)(p_{i-1}+p_i) + DA(c_i^o)^*p_{i-1}\\ &\qquad\qquad - \operatorname{div}((-\Delta)^{-1}p_i\overline{v}_i^o) - \operatorname{div}((-\Delta)^{-1}[D^+c_i^o]q_i)\,,\ c_i^\delta\Big\rangle_{W_1}\\ &\quad+ \left\langle\frac{1}{\tau}(-\Delta)^{-1}p_{M-1} + \frac{1}{2}(-\Delta - I)p_{M-1} + DA(c_M^o)^*p_{M-1}, c_M^\delta\right\rangle_{W_1}.\end{aligned}$$

Hence, we can choose c_i^δ arbitrarily for $i > 0$, which implies the assertion on p. Next, we use $(0, 0, u^\delta) \in C_{(0,0)}$ and find $\left(u^o|u^\delta\right)_{U_1^{M+1}} = \left(-S_+u^\delta|r\right)_{U_1^M}$ and hence $r + S_+u^o = 0$. Finally, choosing $(0, v^\delta, 0) \in C_{(0,0)}$ and proceeding as before yields

$$\begin{aligned}0 &= \big\langle D^+v^\delta - \Delta\overline{v}^\delta + (S_-v^\delta\cdot\nabla)\overline{v}^o + (S_-v^o\cdot\nabla)\overline{v}^\delta + (-\Delta)^{-1}(\nabla S_-c^o\cdot\overline{v}^\delta)\nabla S_-c^o, q\big\rangle_{\widetilde{V}_1^M}\\ &\quad+ \big\langle(-\Delta)^{-1}[\nabla S_-c^o\cdot\overline{v}^\delta], p\big\rangle_{W_1^M} + \left(S_+P_{U_1}\operatorname{Tr}v^\delta|r\right)_{U_1^M}\\ &= \sum_{i=1}^{M-1}\Big\langle\frac{1}{\tau}I^*(q_{i-1}-q_i) - \frac{1}{2}\Delta^*(q_{i-1}+q_i) + b_1(\overline{v}_i^o, q_i) + \frac{1}{2}\Big(b_2(v_{i-1}^o, q_{i-1}) + b_2(v_i^o, q_i)\Big)\\ &\qquad\qquad+\frac{1}{2}\Big((-\Delta)^{-1}(\nabla c_{i-1}^o\cdot q_{i-1})\nabla c_{i-1}^o + (-\Delta)^{-1}(\nabla c_i^o\cdot q_i)\nabla c_i^o\Big)\\ &\qquad\qquad+\frac{1}{2}\Big((-\Delta)^{-1}p_{i-1}\nabla c_{i-1}^o + (-\Delta)^{-1}p_i\nabla c_i^o\Big) - \operatorname{Tr}^*P_{U_1}^*u_i^o\,,\ v_i^\delta\Big\rangle_{V_1}\\ &\quad+\Big\langle\frac{1}{\tau}I^*q_{M-1} + \frac{1}{2}\Big[-\Delta^*q_{M-1} + b_2(v_{M-1}^o, q_{M-1}) + (-\Delta)^{-1}(\nabla c_{M-1}^o\cdot q_{M-1})\nabla c_{M-1}^o\\ &\qquad\qquad+(-\Delta)^{-1}p_{M-1}\nabla c_{M-1}^o\Big] - \operatorname{Tr}^*P_{U_1}^*u_M^o\,,\ v_M^\delta\Big\rangle_{V_1}.\end{aligned}$$

This concludes the proof. ∎

Lemma 3.5. *For every $c \in W_1, v \in V_1$ and every monotone operator $A \subset W_1 \times W_{-1}$ the operators*

$$\begin{aligned}\mathcal{A}_1^A \subset W_1\times W_{-1},&\quad \mathcal{A}_1^A := \tfrac{1}{\tau}(-\Delta)^{-1} + \tfrac{1}{2}(-\Delta - I) + A,\\ \mathcal{A}_2^{c,v}: \widetilde{V}_1 \to \widetilde{V}_{-1},&\quad \mathcal{A}_2^{c,v}q := \tfrac{1}{\tau}I + \tfrac{1}{2}\Big[-\Delta + b_2(v,q) + (-\Delta)^{-1}(\nabla c\cdot q)\nabla c\Big]\end{aligned}$$

are strongly monotone. Moreover, $(0,0) \in \mathcal{A}_2^{c,v}$*, and* $(0,0) \in \mathcal{A}_1^A$ *if* $(0,0) \in A$.

Proof. The strong monotonicity of $\frac{1}{\tau}I - \frac{1}{2}\Delta$ is obvious and the strong monotonicity of $\frac{1}{\tau}(-\Delta)^{-1} + \frac{1}{2}(-\Delta - I)$ is a direct consequence of Lemma 2.10. Consequently, $\mathcal{A}_1^A$ and $\mathcal{A}_2^{c,v}$ are strongly monotone as well, since A is monotone and the remaining terms of $\mathcal{A}_2^{c,v}$ are linear and monotone because of

$$\begin{aligned} \langle b_2(v,q), q\rangle_{V_1} &= 0, \\ \big\langle (-\Delta)^{-1}(\nabla c \cdot q)\nabla c, q\big\rangle_{V_1} &= \big\langle \nabla c \cdot q, (-\Delta)^{-1}(\nabla c \cdot q)\big\rangle_{W_1} \geq 0. \end{aligned}$$

This concludes the proof. ∎

Theorem 3.6. *Let* $(\varphi^{(n)})_{n\in\mathbb{N}^*}$ *be a sequence of proper, convex and lower-semicontinuous functionals from* W_1 *into* $\overline{\mathbb{R}}$ *satisfying* (H$_1$) *for a common constant* $C = C_{\varphi^{(n)}}$ *for all* $n \geq 1$ *as well as* (H$_2$) *or* (H$_3$)*. Moreover,* $A^{(n)} = \partial\varphi^{(n)}$ *is assumed to be single-valued and continuously Fréchet differentiable on* W_1 *for* $n \in \mathbb{N}$*. Suppose that* $(c^{(n)}, v^{(n)}, u^{(n)}) \in W_1^{M+1} \times V_1^{M+1} \times U_1^{M+1}$ *are minimizers for* (P$_{\varphi^{(n)}}$) *for all* $n \in \mathbb{N}$ *with* $(J(c^{(n)}, v^{(n)}, u^{(n)}))$ *being bounded. Then there exist* $(p^{(n)}, q^{(n)}) \in W_1^M \times \widetilde{V}_1^M$ *satisfying*

$$\begin{aligned} 0 &= -(-\Delta)^{-1}D^+p^{(n)} + (-\Delta - I)\overline{p}^{(n)} + DA^{(n)}(S_+c^{(n)})^*S_-p^{(n)} \qquad (3.6)\\ &\quad - \operatorname{div}((-\Delta)^{-1}(S_+p^{(n)})S_+\overline{v}^{(n)}) - \operatorname{div}((-\Delta)^{-1}D^+(S_+c^{(n)})S_+q^{(n)}), \\ \gamma(c_M^{(n)} - c_e) &= \Big(\frac{1}{\tau}(-\Delta)^{-1} + \frac{1}{2}(-\Delta - I) + DA^{(n)}(c_M^{(n)})^*\Big)p_{M-1}^{(n)}, \qquad (3.7) \end{aligned}$$

$$\begin{aligned} S_+\, Tr^* P_{U_1}^* u^{(n)} &= -I^*D^+q^{(n)} - \Delta^*\overline{q}^{(n)} + b_1(S_+\overline{v}^{(n)}, S_+q^{(n)}) \qquad (3.8)\\ &\quad + \frac{1}{2}\Big(b_2(S_-v^{(n)}, S_-q^{(n)}) + b_2(S_+v^{(n)}, S_+q^{(n)})\Big) \\ &\quad + \frac{1}{2}\Big((-\Delta)^{-1}(\nabla S_-c^{(n)} \cdot S_-q^{(n)})\nabla S_-c^{(n)} \\ &\qquad + (-\Delta)^{-1}(\nabla S_+c^{(n)} \cdot S_+q^{(n)})\nabla S_+c^{(n)}\Big) \\ &\quad + \frac{1}{2}\Big((-\Delta)^{-1}(S_-p^{(n)})\nabla S_-c^{(n)} + (-\Delta)^{-1}(S_+p^{(n)})\nabla S_+c^{(n)}\Big), \\ Tr^* P_{U_1}^* u_M^{(n)} &= \frac{1}{\tau}I^*q_{M-1}^{(n)} + \frac{1}{2}\Big[-\Delta^*q_{M-1}^{(n)} + b_2(v_{M-1}^{(n)}, q_{M-1}^{(n)}) \qquad (3.9)\\ &\qquad + (-\Delta)^{-1}(\nabla c_{M-1}^{(n)} \cdot q_{M-1}^{(n)})\nabla c_{M-1}^{(n)} + (-\Delta)^{-1}p_{M-1}^{(n)}\nabla c_{M-1}^{(n)}\Big], \end{aligned}$$

where last two equations are understood in the sense of $(Z^*)^{M-1}$ *and* Z^* *for* $Z := \{v \in V_1 : P_{U_1^\perp}\, Tr\, v = 0\}$*. For a subsequence (denoted by index* m*) it holds that*

$$\begin{aligned} c^{(m)} &\to c^{(\infty)} && \text{in } W_1^{M+1}, \\ (p^{(m)}, q^{(m)}, v^{(m)}, u^{(m)}) &\rightharpoonup (p^{(\infty)}, q^{(\infty)}, v^{(\infty)}, u^{(\infty)}) && \text{in } W_1^M \times V_1^M \times V_1^{M+1} \times U_1^{M+1}, \\ DA^{(n)}(S_+c^{(m)})^*S_-p^{(m)} &\rightharpoonup \lambda^{(\infty)} && \text{in } W_{-1}^{M-1}. \end{aligned}$$

Moreover, $(c^{(\infty)}, v^{(\infty)}, u^{(\infty)})$ *is a minimizer of* $(\mathrm{P}_{\varphi^{(\infty)}})$ *and we have that*

$$\begin{aligned} 0 &= -(-\Delta)^{-1} D^+ p^{(\infty)} + (-\Delta - I)\overline{p}^{(\infty)} + \lambda^{(\infty)} \\ &\quad - \operatorname{div}((-\Delta)^{-1}(S_+ p^{(\infty)}) S_+ \overline{v}^{(\infty)}) - \operatorname{div}((-\Delta)^{-1} D^+(S_+ c^{(\infty)}) S_+ q^{(\infty)}), \end{aligned} \tag{3.10}$$

$$\gamma(c_M^{(\infty)} - c_e) = \Big(\frac{1}{\tau}(-\Delta)^{-1} + \frac{1}{2}(-\Delta - I)\Big) p_{M-1}^{(\infty)} + \lambda_{M-1}^{(\infty)}, \tag{3.11}$$

$$\begin{aligned} S_+ \, Tr^* P_{U_1}^* u^{(\infty)} &= -I^* D^+ q^{(\infty)} - \Delta^* \overline{q}^{(\infty)} + b_1(S_+ \overline{v}^{(\infty)}, S_+ q^{(\infty)}) \\ &\quad + \frac{1}{2}\Big(b_2(S_- v^{(\infty)}, S_- q^{(\infty)}) + b_2(S_+ v^{(\infty)}, S_+ q^{(\infty)})\Big) \\ &\quad + \frac{1}{2}\Big((-\Delta)^{-1}(\nabla S_- c^{(\infty)} \cdot S_- q^{(\infty)}) \nabla S_- c^{(\infty)} \\ &\qquad + (-\Delta)^{-1}(\nabla S_+ c^{(\infty)} \cdot S_+ q^{(\infty)}) \nabla S_+ c^{(\infty)} \Big) \\ &\quad + \frac{1}{2}\Big((-\Delta)^{-1}(S_- p^{(\infty)}) \nabla S_- c^{(\infty)} + (-\Delta)^{-1}(S_+ p^{(\infty)}) \nabla S_+ c^{(\infty)} \Big), \end{aligned} \tag{3.12}$$

$$\begin{aligned} Tr^* P_{U_1}^* u_M^{(\infty)} &= \frac{1}{\tau} I^* q_{M-1}^{(\infty)} + \frac{1}{2}\Big[-\Delta^* q_{M-1}^{(\infty)} + b_2(v_{M-1}^{(\infty)}, q_{M-1}^{(\infty)}) \\ &\qquad + (-\Delta)^{-1}(\nabla c_{M-1}^{(\infty)} \cdot q_{M-1}^{(\infty)}) \nabla c_{M-1}^{(\infty)} + (-\Delta)^{-1} p_{M-1}^{(\infty)} \nabla c_{M-1}^{(\infty)} \Big], \end{aligned} \tag{3.13}$$

Proof. We split the proof into three steps. We first prove the strong convergence of a subsequence $(c^{(m)}, w^{(m)}, v^{(m)})$ to a minimizer of $(\mathrm{P}_{\varphi^{(\infty)}})$ and then we establish the weak convergence of the adjoint state. Finally, we pass to the limit in the first order system.

1. Using Theorem 3.4 we find sequences $(p^{(n)})_{n\geq 1}$ in W_1^M and $(q^{(n)})_{n\geq 1}$ in V_1^M satisfying equations (3.6)–(3.8). Moreover, the coercivity of J in u and the boundedness of $(J(c^{(n)}, u^{(n)}))$ imply that $(u^{(n)})$ is bounded in U_1^{M+1}. Taking advantage of Proposition 2.14 it follows that we can pass to subsequences $(c^{(m)})$ and $(v^{(m)}, u^{(m)})$ that converge strongly to $c^{(\infty)}$ in W_1^{M+1} and weakly to $(v^{(\infty)}, u^{(\infty)})$ in $V_1^{M+1} \times U_1^{M+1}$, respectively, with $(c^{(\infty)}, v^{(\infty)}, u^{(\infty)})$ being a minimizer of $(\mathrm{P}_{\varphi^{(\infty)}})$.

2. Now we show that $(p^{(n)})$ and $(q^{(n)})$ are bounded in W_1^M respectively $\widetilde{V}_1^M$. This is done by induction. First observe that the boundedness of $(c_M^{(n)} - c_e)$ together with Lemma 3.5 and the fact that $(0,0) \in DA^{(n)}(c_M^{(n)})^*$ show that $(p_{M-1}^{(n)})$ is bounded in W_1. This together with (3.9) and the second strong monotonicity provided in Lemma 3.5 yields the boundedness of $(q_{M-1}^{(n)})$ in $\widetilde{V}_1$. Next, assume that $(p_{i+1}^{(n)})$ and $(q_{i+1}^{(n)})$ are bounded for some $0 \leq i < M-1$. Using the definitions of D^+x and of $\overline{x}$ we obtain from (3.6) and (3.8) that

$$\begin{aligned} \mathcal{A}_1 p_i^{(n)} &= \big[\tfrac{1}{\tau}(-\Delta)^{-1} - \tfrac{1}{2}(-\Delta - I)\big] p_{i+1}^{(n)} + \operatorname{div}((-\Delta)^{-1} p_{i+1}^{(n)} \overline{v}_i^{(n)}) \\ &\quad + \operatorname{div}((-\Delta)^{-1}(D^+ c^{(n)})_{i+1} q_{i+1}^{(n)}), \\ \mathcal{A}_2 q_i^{(n)} &= \tfrac{1}{\tau} q_{i+1}^{(n)} - b_1(\overline{v}_{i+1}^{(n)}, q_{i+1}^{(n)}) - \frac{1}{2}\Big[-\Delta q_{i+1}^{(n)} + b_2(v_{i+1}^{(n)}, q_{i+1}^{(n)}) \\ &\quad + (-\Delta)^{-1} \nabla c_{i+1}^{(n)} \cdot q_{i+1}^{(n)} \nabla c_{i+1}^{(n)} + (-\Delta)^{-1} p_i^{(n)} \nabla c_i^{(n)} + (-\Delta)^{-1} p_{i+1}^{(n)} \nabla c_{i+1}^{(n)} \Big] \\ &\quad - \mathrm{Tr}^* \, P_{U_1}^* u_{i+1}^{(n)}, \end{aligned}$$

where $\mathcal{A}_1 := \mathcal{A}_1^{DA^{(n)}(c_{i+1}^{(n)})^*}$ and $\mathcal{A}_2 := \mathcal{A}_2^{c_i^{(n)}, v_i^{(n)}}$ are given as in Lemma 3.5. Applying Lemma 3.5 to the first and afterwards to the second equation, we conclude the boundedness of $(p_i^{(n)})$ and $(q_i^{(n)})$ in W_1 respectively $\widetilde{V}_1$.

3. By (3.6), also $DA^{(n)}(S_+c^{(n)})^*S_-p^{(n)}$ remains bounded in W_{-1}^M. Therefore, we pass to subsequences (denote by index m again) to obtain the desired convergence result for $(p^{(m)})$, $(q^{(m)})$ and $DA^{(n)}(S_+c^{(m)})^*S_-p^{(m)}$. Using the strong and weak convergences, the properties of the operators involved and passing to the limit in (3.6)–(3.9) as $m \to \infty$ we finally end up with (3.10)–(3.13). □

4. Application to the double-obstacle potential. In this section we consider the case where φ is given by the indicator function of a special convex subset of W_1. This corresponds to the Cahn-Hilliard system with double-obstacle potential. Moreover, the $\varphi^{(n)}$ are defined as mollified versions of the Moreau-Yosida approximations in W_0 of $\varphi := \varphi^{(\infty)}$. In this setting, the optimization problem (P_φ) becomes a mathematical program with complementarity constraints since (2.2) indeed becomes a variational inequality. In this context our approach yields a function space version of C-stationarity; see [19] for the latter.

Double-obstacle potential: Let $k_1, k_2 \in \mathbb{R}$ with $k_1 < 0 < k_2$. We define

$$K := [k_1, k_2], \qquad \psi := \imath_K : \mathbb{R} \to \overline{\mathbb{R}}, \qquad \theta := \partial\varphi \subset \mathbb{R} \times \mathbb{R},$$
$$K_0 := \{c \in W_0 : c(x) \in K \text{ for a.e. } x \in \Omega\}, \quad K_1 := K_0 \cap W_1.$$

Then $\varphi := \imath_{K_1} : W_1 \to \overline{\mathbb{R}}$ defines a so-called double-obstacle potential.

Moreover, let $\rho \in C^1(\mathbb{R})$ denote a fixed mollifier with $\operatorname{supp}\rho \subset [-1,1]$, $\int_{\mathbb{R}} \rho = 1$ and $0 \le \rho \le 1$ a.e. on Ω, and $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ be a function with $\varepsilon(\alpha) > 0$ and $\frac{\varepsilon(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$. We consider the Yosida approximation θ_α (with parameter $\alpha > 0$) of θ (for the general definition we refer to [3]) and define

$$\rho_\varepsilon(r) := \frac{1}{\varepsilon}\rho\Big(\frac{r}{\varepsilon}\Big), \qquad \beta_\alpha := \theta_\alpha * \rho_{\varepsilon(\alpha)}, \qquad \widetilde{\theta}_\alpha(r) := \int_0^r \beta_\alpha, \qquad \varphi_\alpha(c) := \int_\Omega \widetilde{\theta}_\alpha \circ c,$$

where $*$ denotes the usual convolution operator. Let $\tau_1 > 0$ be fixed such that the sequence of functionals $(\varphi^{(n)})_{n\in\mathbb{N}^*}$ defined by

$$\varphi^{(\infty)} := \varphi, \qquad \alpha_n := \tau_1 n^{-1}, \qquad \varphi^{(n)} := \varphi_{\alpha_n}$$

satisfies the conditions (H_3) and (H_1) for some common constant C.

In what follows we collect a few useful properties of φ and its approximations $\varphi^{(n)}$. For proofs and further details we refer to [21].

Remark 4.1.

1. The mapping $\beta_\alpha : \mathbb{R} \to \mathbb{R}$ is a regularization of the Yosida approximation θ_α of θ and $\varphi_\alpha : W_1 \to \mathbb{R}$ a regularization of the Moreau-Yosida approximation of φ in $L^2(\Omega)$.

2. The existence of $\tau_1 > 0$ such that $(\varphi^{(n)})_{n\in\mathbb{N}^*}$ satisfies conditions (H_3) and (H_1) for some common constant C was shown in Proposition 4.3 of [21].

3. The assumption $\rho \in C^2(\mathbb{R})$ in [21] is only necessary to guarantee the additional regularity $\beta_\alpha \in C^2(\mathbb{R})$, which is not needed for our purpose.

4. For sufficiently small α, β_α vanishes identically in a neighborhood of 0. Note further that we could choose different mollifiers ρ^1 and ρ^2 instead of ρ in the definition of $\beta_\alpha(s)$ for either positive s or negative s, respectively. Thus, conditions (H_1) and (H_3) remain true also in this case.

THEOREM 4.1. *Consider the setting of this section and suppose that* $h : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz function with* $h(k_1) = h(k_2) = 0$. *Then the assumptions of Theorem 3.6 are fulfilled and for the sequences* $(a^{(n)})$ *and* $(\lambda^{(n)})$ *given by* $a_0^{(n)} = 0$ *and*

$$S_+a^{(n)} := (-\Delta)^{-1}f^c - [(-\Delta)^{-1}D^+c^{(n)} + (-\Delta - I)\overline{c}^{(n)} + (-\Delta)^{-1}\nabla S_-c^{(n)} \cdot \overline{v}^{(n)}],$$
$$\lambda^{(n)} := DA^{(n)}(S_+c^{(n)})^*p^{(n)},$$

and for a subsequence denoted by index m *and all* $i = 0, ..., M-1$ *it holds that*

$$\big(S_+a^{(\infty)} \,|\, S_+c^{(\infty)} - k_1 \big)_{W_0} = 0, \big(S_+a^{(\infty)} \,|\, S_+c^{(\infty)} - k_2 \big)_{W_0} = 0,$$
$$\lim \big(\lambda^{(m)} \,|\, h(S_+c^{(m)}) \big)_{W_0} = 0, \lim \big(S_+a^{(m)} \,|\, p^{(m)} \big)_{W_0} = 0,$$
$$\underline{\lim} \big(\lambda^{(m)} \,|\, p^{(m)} \big)_{W_0} \geq 0,$$

$$\lambda_i^{(m)} \to 0 \text{ almost everywhere on } \{x \in \Omega : k_1 < c_i(x) < k_2\}.$$

Proof. 0. Since the double obstacle potential φ satisfies the conditions (H$_3$) and (H$_1$) for a common constant and since all $A^{(n)} = \partial\varphi^{(n)}$ are single-valued and continuously Fréchet differentiable on W_1, it remains to show that $(J(c^{(n)}, v^{(n)}u^{(n)}))$ is bounded for a sequence of minimizers $(c^{(n)}v^{(n)}u^{(n)})$ as given in Theorem 3.6. But since $(c, v) = (0, 0)$ is a solution to $\mathcal{S}_{\varphi^{(n)}}u$ for $u = 0$ and every $n \in \mathbb{N}$, it follows that $0 \leq J(c^{(n)}, v^{(n)}u^{(n)}) \leq \frac{1}{2}\gamma\|c_e\|^2_{W_0}$ for all $n \in \mathbb{N}$.

1. We start by showing the complementarity conditions $\big(S_+a^{(\infty)} \,|\, S_+c^{(\infty)} - k_1 \big)_{W_0} = 0$ and $\big(S_+a^{(\infty)} \,|\, S_+c^{(\infty)} - k_2 \big)_{W_0} = 0$. Since $(S_+c^{(\infty)}, S_+a^{(\infty)}) \in A$ by Proposition 3.1 and since A is the superposition operator of $\theta = \partial\imath_K \subset \mathbb{R} \times \mathbb{R}$, we conclude that $(c_i^{(\infty)}, a_i^{(\infty)}) \in \theta$ for almost all $x \in \Omega$ and $i > 0$ and therefore $a_i^{(\infty)}(x)[c_i^{(\infty)}(x) - k_1] = 0$ and $a_i^{(\infty)}(x)[c_i^{(\infty)}(x) - k_2] = 0$. Integration yields the complementarity conditions.

2. Next, we prove $\lim \big(\lambda^{(m)} \,|\, h(S_+c^{(m)}) \big)_{W_0} = 0$. Denoting the metric projection of $\mathbb{R}$ onto $K = [k_1, k_2]$ by p_K and the metric projection of W_0 onto K_0 by P (which is the superposition operator of p_K), respectively, and taking advantage of the continuity of the superposition operator of h on W_1 (cf. [29]), it follows that $P(W_1) \subset W_1$ and $\lim Pc^{(m)} = Pc^{(\infty)} = c^{(\infty)}$, $\lim h(Pc^{(m)}) = h(Pc^{(\infty)}) = h(c^{(\infty)}) = \lim h(c^{(m)})$ in $(H^1(\Omega))^{M+1}$. Moreover, it holds that $|\beta'_\alpha(s)| \leq \frac{1}{\alpha}$ for all s and $\beta'_\alpha(s) = 0$ for $k_1 + \varepsilon(\alpha) \leq s \leq k_2 - \varepsilon(\alpha)$; see [21] for details. If L_h is the Lipschitz constant of h, then $|h(s)| \leq L_h \min(|s - k_1|, |s - k_2|)$ for $s \in \mathbb{R}$. Consequently, it follows that

$$\begin{aligned} |\big(\lambda^{(m)} \,|\, h(PS_+c^{(m)}) \big)_{W_0}|^2 &= |\big(p^{(m)} \,|\, DA^{(m)}(S_+c^{(m)})h(PS_+c^{(m)}) \big)_{W_0}|^2 \\ &\leq \|\, p^{(m)} \,\|^2_{W_0} \sum_{i=1}^{M} \int_\Omega |\beta'_{\alpha_m}(c_i^{(m)})h(Pc_i^{(m)})|^2 \\ &\leq \left(M|\Omega| \,\|\, p^{(m)} \,\|_{W_0} \, L_h \frac{\varepsilon(\alpha_m)}{\alpha_m} \right)^2 \;\to\; 0 \end{aligned}$$

as $m \to \infty$. Moreover, since $(\lambda^{(m)})$ is bounded in W_{-1}^M we have that

$$\begin{aligned} &\lim \big(\lambda^{(m)} \,|\, h(S_+c^{(m)}) \big)_{W_0} \\ &\quad = \lim \big(\lambda^{(m)} \,|\, h(PS_+c^{(m)}) \big)_{W_0} + \lim \Big\langle \lambda^{(m)}, h(S_+c^{(m)}) - h(PS_+c^{(m)}) \Big\rangle_{W_1^M} \\ &\quad = 0. \end{aligned}$$

3. We set $g_m(s) := \beta_{\alpha_m}(s) - \beta'_{\alpha_m}(s)\kappa(s)$ with $s - p_K(s) =: \kappa(s)$. Then we obtain

$$\begin{aligned}\big(S_+a^{(m)} \,|\, p^{(m)}\big)_{W_0} &= \big(p^{(m)} \,|\, \beta_{\alpha_m}(S_+c^{(m)})\big)_{W_0} \\ &= \big(p^{(m)} \,|\, g_m(S_+c^{(m)})\big)_{W_0} + \big(\lambda^{(m)} \,|\, S_+c^{(m)} - PS_+c^{(m)}\big)_{W_0}.\end{aligned}$$

By Lemma 4.2 in [21], for m sufficiently large it holds that $|g_m(s)| = |\beta_{\alpha_m}(s) - \beta'_{\alpha_m}(s)\kappa(s)| \leq C\,\frac{\varepsilon(\alpha_m)}{\alpha_m}$. Hence, the first term on the right hand side converges to 0. This is also true for the second term since $(\lambda^{(m)})$ is bounded in W_{-1}^M and $(c^{(m)})$ and $(Pc^{(m)})$ both converge to $c^{(\infty)}$ in W_1^{M+1}.

4. The fact that $\underline{\lim}\,\big(\lambda^{(m)} \,|\, p^{(m)}\big)_{W_0} \geq 0$ is a consequence of the monotonicity of $DA^{(n)}(c) : W_1 \to W_{-1}$ for every $c \in W_1$. Indeed, given $\overline{c} \in W_1$ we have

$$\Big\langle DA^{(n)}(c)\overline{c}, \overline{c}\Big\rangle_{W_1} = \lim_{t \to 0} \frac{1}{t^2}\Big\langle A^{(n)}(c + t\overline{c}) - A^{(n)}c, (c + t\overline{c}) - c\Big\rangle_{W_1} \geq 0$$

by the monotonicity of $A^{(n)}$.

5. Let us fix $i \in \{0, ..., M-1\}$ and representatives of the equivalence classes $c^{(\infty)}, (c^{(m)})$. Further, define $Z := \{x \in \Omega : k_1 < c_i^{(\infty)}(x) < k_2\}$. Since $c_i^{(m)}$ converges to $c_i^{(\infty)}$ in W_1, a subsequence converges almost everywhere on Ω. Without loss of generality, we assume that $(c^{(m)})$ itself has this convergence property. Moreover, we know that $\varepsilon(\alpha_m) \to 0$. Hence, for almost all $x \in Z$ there exists $m_0(x)$ such that $k_1 + \varepsilon(\alpha_m) < c^{(m)}(x) < k_2 - \varepsilon(\alpha_m)$ for all $m \geq m_0(x)$.
From the properties of β_α it follows that $\lambda^{(m)}(x) = 0$ for almost all $x \in Z$ and $m \geq m_0(x)$. Consequently, $\lambda^{(m)}$ converges to 0 almost everywhere on Z. ∎

We remark that compared to weaker forms of stationarity, for instance, as those contained in [4] for certain classes of optimal control problems for variational inequalities, C-stationarity represents a sharper stationarity notion avoiding spurious stationarity points. A numerical realization based on an extension of the algorithms in [20] to the present Cahn-Hilliard/Navier-Stokes-setting will be the subject of future work.

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