fully inert subgroups of Abelian groups

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While studying the so-called *intrinsic algebraic entropy* for endomorphisms of Abelian groups, Dikranjan, Giordano Bruno, Salce, Virili (JPAA, to appear) where led to introduce the notion of fully inert subgroup of an Abelian group G.
Fully inert subgroups are naturally defined objects, that present many interesting features and deserve to be studied independently.

DEFINITION

All groups are assumed to be Abelian.

A subgroup H of (an Abelian group) G is called *fully inert* if $(\phi H + H)/H$ is finite for every $\phi \in \mathrm{End}(G)$.

Finite and finite-index subgroups of G are fully inert. If $L \subseteq G$ is *fully invariant* (i.e., $\phi L \subseteq L$ for every $\phi \in \mathrm{End}(G)$), then L is obviously fully inert.

Our aim is to compare the fully inert subgroups of G with the fully invariant ones, which, in several cases, are well understood.
We don't describe other technical characterizations of fully inert subgroups.

Dikranjan - Giordano Bruno - Salce - Virili, Fully Inert Subgroups of divisible Abelian groups, J. Group Theory 16, 2013.
Such groups extend the classical notion of *quasi-injective* Abelian groups. We don't discuss the results of this paper.

Dikranjan - Salce - PZ, Fully inert subgroups of free Abelian groups, Periodica Math. Hung., 2014.

Goldsmith - Salce - PZ, Fully inert submodules of torsion-free modules over the ring of p-adic integers, Colloquium Math. 136(2), 2014.

Goldsmith - Salce - PZ, Fully inert subgroups of Abelian p-groups, J. Algebra 419, 2014.

Let A, B be subgroups of G. A is *commensurable* with B if both $(A + B)/A$ and $(A + B)/B$ are finite.

Commensurability is an equivalence, [DSZ, Per. Math. Hung.]. Commensurable subgroups are "close".

Proposition [DGSV, J. Group Th. 2013]

If $A, B \subseteq G$ are commensurable, and A is fully inert, then also B is fully inert. In particular, if L is fully invariant in G, and H is commensurable with L, then H is fully inert.

Question. Determine the classes of groups G such that: H fully inert in G implies H commensurable with a fully invariant subgroup.

Results proved in [DSZ], Per. Math. Hung. 2014.

Lemma

Let G be a free group, H a fully inert subgroup of G. Then G/H is bounded (i.e., $m(G/H) = 0$ for some $m > 0$).

Theorem

A subgroup H of a free group G is fully inert if and only if it is commensurable with a fully invariant subgroup of G, that is, with nG for some integer $n \geq 0$.

Here $J_p = \hat{\mathbb{Z}}_p$ denotes the ring of p-adic integers. Results proved in [GSZ], Colloquium Math. 2014.

THEOREM

A submodule H of a complete torsion-free J_p-module $\hat{A}$ is fully inert if and only if it is commensurable with a fully invariant submodule, that is with $p^n\hat{A}$, for some $n \geq 0$.

THEOREM

There exist torsion-free J_p-modules X that contain fully inert submodules not commensurable with any fully invariant submodule.

X is constructed using realization theorems of J_p-algebras proved by Goldsmith (after Corner 1960s fundamental theorems).

Results proved in [GSZ], J. Algebra 2014.

Let $G = \bigoplus_{0<n<\kappa} G_n$ be a direct sum of cyclic p-groups, where $\kappa \leq \omega$, and G_n is a *nonzero* direct sum of copies of $\mathbb{Z}/p^{c_n}\mathbb{Z}$, where $0 < c_1 < \cdots < c_n < \ldots$.

LEMMA (BENABDALLAH - EISENSTADT - IRWIN - POLUIANOV, ACTA MATH. HUNGAR. 1970)

Let $G = \bigoplus_{0<n<\kappa} G_n$ be as above. Then L is a fully invariant subgroup of G if and only if $L = \bigoplus_{0<n<\kappa} p^{h(n)} G_n$, where the integers $h(n)$ satisfy the conditions
(1) $h(n) \leq c_n$ for all $n > 0$;
(2) $h(i) \leq h(n) \leq h(i) + c_n - c_i$ for all $0 < i < n$.

THEOREM

Let H be a fully inert subgroup of a bounded p-group G (i.e., κ is finite). Then H is commensurable with a fully invariant subgroup of G.

The proof requires a crucial lemma and six steps!
(direct proof)

For the general case we need some structural arguments.
Let H be an arbitrary subgroup of a p-group G. We denote by H^* the intersection of the fully invariant subgroups of G containing H. We call it the *fully invariant hull* of H.
Let now $G = \bigoplus_{0<n<\kappa} G_n$ be an unbounded direct sum of cyclic p-groups (i.e., $\kappa = \omega$).
For each $t < \omega$, let $G^t = \bigoplus_{n \geq t} G_n$. For H any subgroup of G, define $H^t = H \cap G^t$ and denote by H^{*t} the fully invariant hull of H^t in G^t.

Crucial result to get the main theorem.

Theorem

Let H be a fully inert subgroup of the direct sum of cyclic p-groups G. Then there exists $t > 0$ such that $(H^{*t} + H)/H$ is finite.

Main Theorem

A fully inert subgroup H of a direct sum of cyclic p-groups G is commensurable with a fully invariant subgroup of G.

Theorem

There exist special p-groups G (constructed by Pierce, 1963) that contain fully inert subgroups not commensurable with any fully invariant subgroup.