Sets of positive integers closed under product and the number of decimal digits

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- Introduction
- LD-semigroups
- Frobenius variety of LD-semigroups
- The smallest digital semigroup containing a set of positive integers
- The smallest LD-semigroup containing a set of positive integers

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- $L(A) = \{\ell(a) \mid a \in A\}$, for A a subset of $\mathbb{N}\backslash\{0\}$.

LD-semigroups

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Remark

Observe that not every numerical semigroup is a LD-semigroup.
For example, $S = < 4, 5 >$ is not a LD-semigroup, because $2 \times 4 - 1 \notin S$.

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1) *S is a LD-semigroup.*
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Let $\mathcal{D} = \{D \mid D \text{ is a digital semigroup}\}$ and let $\mathcal{L} = \{S \mid S \text{ is a LD-semigroup}\}$.

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Corollary

The correspondence $\varphi : \mathcal{D} \to \mathcal{L}$, defined by $\varphi(D) = L(D) \cup \{0\}$, is a bijective map. Furthermore its inverse is the map $\theta : \mathcal{L} \to \mathcal{D}$,
$\theta(S) = \{a \in \mathbb{N} \setminus \{0\} \mid \ell(a) \in S\}$.
So,

$$\mathcal{D} = \{\theta(S) \mid S \text{ is a LD-semigroup}\}.$$

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A Frobenius variety is a nonempty set $\mathcal{V}$ of numerical semigroups fulfilling the following conditions:

1. if S and T are in $\mathcal{V}$, then so is $S \cap T$;
2. if S is in $\mathcal{V}$ and it is not equal to $\mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is in $\mathcal{V}$.

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Proposition

Let $\mathcal{L} = \{S \mid S \text{ is a LD-semigroup}\}$. The set $\mathcal{L}$ is a Frobenius variety.

A graph is a pair $G = (V, E)$, where V is a nonempty set and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$. The elements of V are called vertices of G and the elements of E are its edges. A path of length n connecting the vertices x and y of G is a sequence of distinct edges of the form (v_0, v_1), (v_1, v_2),. . .,(v_{n-1}, v_n) such that $v_0 = x$ and $v_n = y$. A graph $G = (V, E)$ is a tree if there exists a vertex r (known as the root of G) such that for every other vertex x there exist a unique path connecting x and r. If (x, y) is an edge of a tree, then we say that x is a son of y.

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Let $\mathcal{L} = \{S \mid S \text{ is a LD-semigroup}\}$. We define the graph $G(\mathcal{L})$ as the graph whose vertices are the elements of $\mathcal{L}$ and $(S, S') \in \mathcal{L} \times \mathcal{L}$ is an edge if $S' = S \cup \{\mathrm{F}(S)\}$.

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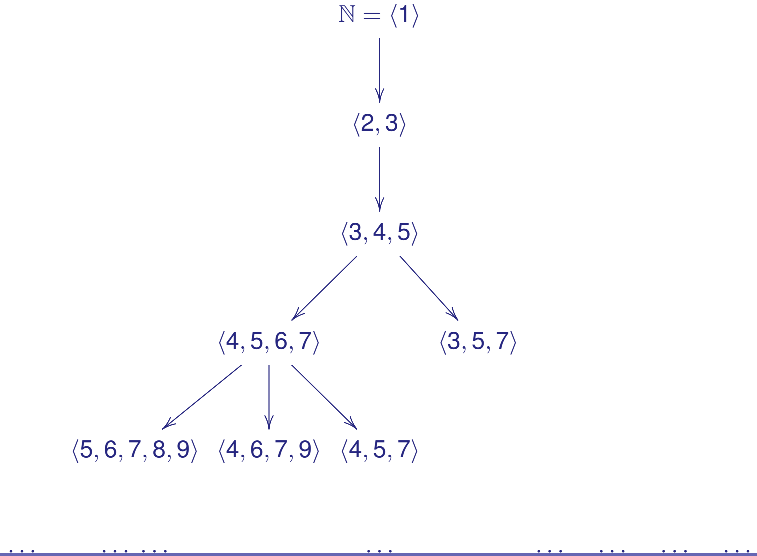
Theorem

The graph $G(\mathcal{L})$ is a tree rooted in $\mathbb{N}$. Moreover, the sons of a vertex $S \in \mathcal{L}$ are $S \setminus \{x_1\}, \ldots, S \setminus \{x_l\}$ with
$\{x_1, \ldots, x_l\} = \{x \in \mathrm{msg}(S) \mid x > \mathrm{F}(S) \text{ and } S \setminus \{x\} \in \mathcal{L}\}$

Frobenius variety of LD-semigroups

Figure: The tree of LD-numerical semigroups



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Given $A \subseteq \mathbb{N}$, we denote by $\mathcal{L}(A)$ the intersection of all LD-semigroups containing A. It is clear that $\mathcal{L}(A)$ is a submonoid of $(\mathbb{N}, +)$

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Proposition

Let A be a nonempty subset of $\mathbb{N} \backslash \{0\}$. Then $\mathcal{L}(A)$ is a LD-semigroup. Moreover, $\mathcal{L}(A)$ is the smallest LD-semigroup containing A.

The smallest digital semigroup containing a set of positive integers

Proposition

Let X be a nonempty subset of $\mathbb{N}\setminus\{0\}$. Then S is the smallest LD-semigroup containing $L(X)$ if and only if $\theta(S)$ is the smallest digital semigroup containing X.

Given a positive integer n, we denote by $\triangle(n) = \{x \in \mathbb{N}\setminus\{0\} \mid \ell(x) = n\}$.

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Example

Let us compute the smallest digital semigroup that contains $\{1235, 54321\}$
The smallest LD-semigroup that contains $L(1235, 54321) = \{4, 5\}$ must contain the number 7 ($2x - 1 \in S$)
So, $< 4, 5, 7 >$ is a LD-semigroup.
Hence $\mathcal{L}(\{4, 5\}) =< 4, 5, 7 >$.
Applying the previous result, we get that
$\mathcal{D}(\{1235, 543221\}) = \theta(< 4, 5, 7 >) = \mathbb{N}\setminus(\triangle(1) \cup \triangle(2) \cup \triangle(3) \cup \triangle(6) \cup \{0\})$

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Proposition

Let S be a numerical semigroup minimally generated by $\{n_1, \ldots, n_p\}$. The following conditions are equivalent:

1. *S is a LD-semigroup;*
2. *if $i, j \in \{1, \ldots, p\}$ then $n_i + n_j - 1 \in S$;*

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algorithm

Input: A set of positive integers A.
Output: The minimal system of generators of the smallest LD-semigroup containing the set A.

1) $B = \mathrm{msg}(\langle A \rangle)$
2) *if $a + b - 1 \in \langle B \rangle$ for all $a, b \in B$, then return B.*
3) $A = B \cup \{a + b - 1 \mid a, b \in B \text{ and } a + b - 1 \notin \langle B \rangle\}$ *and go to 1).*

Example

Let us compute the smallest LD-semigroup that contain $\{5\}$. To this end we use the previous algorithm. The values arising for A and B are:

- $A = \{5\}$;
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- $A = \{5, 9\}$;
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- $A = \{5, 9, 13, 17\}$;
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- $A = \{5, 9, 13, 17, 21\}$;
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Therefore the smallest LD-semigroup containing $\{5\}$ is $\langle 5, 9, 13, 17, 21\rangle = \{0, 5, 9, 10, 13, 14, 15, 17, \rightarrow\}$.

Thank you!