On an extension of the classical zero divisor graph

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Joint work with D. Bennis and J. Mikram

Outline

Motivation and Preliminaries

Notations

- Throughout this talk all rings are commutative with identity element.
- $Z(R)$ denotes the set of zero divisors of R, in particular $Z(R)^* := Z(R)\backslash\{0\}$.
- $Ann(x)$ denotes the annihilator of an element x of R.
- For an ideal I of R, $\sqrt{I}$ means the radical of I, in particular, $Nil(R) := \sqrt{0}$ is the nilradical of R.
- The ring $\mathbb{Z}/n\mathbb{Z}$ of the residues modulo an integer n will be noted by $\mathbb{Z}_n$.
- $T(R) = S^{-1}R$, where S is the set of regular elements, is the total quotient ring of R.
- For a non-zero nilpotent element x of R, n_x denotes the index of nilpotency of x.

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MOTIVATION AND PRELIMINARIES

We recall some basic notions on graph theory.

DEFINITION

- Let G be a (undirected) graph. We say that G is connected if there is a path between any two distinct vertices.
- $d(x, y)$ denotes the distance between x and y in G, is the length of a shortest path connecting x and y and if no such path exists, we set $d(x, y) = \infty$ (by convention $d(x, x) = 0$).
- The diameter of the graph G is the quantity $diam(G) := sup\{d(x, y)|x$ and y are vertices of $G\}$.
- A cycle of length $n \in \mathbb{N}^*$ in G is a path of the form $x_1 - x_2 - \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$.

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- The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G, provided G contains a cycle, otherwise, $gr(G) = \infty$.
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- In 1998, Anderson and Livingston introduced the zero divisor graph of a commutative ring and started the study of the relation between ring-theoretic properties and graph theoretic ones.

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The zero-divisor graph of a ring R, denoted by $\Gamma(R)$, is the simple graph associated to R such that its vertex set consists of all its non-zero zero divisors and that two distinct vertices are joined by an edge if and only if the product of these two vertices is zero.

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- It was proved, among other things, that $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$ and $gr(\Gamma(R)) \in \{3, 4, \infty\}$.
- Since then, the zero divisor graphs of commutative rings have attracted the attention of several researchers, among them Akbari; D.D. Anderson; D.F. Anderson; Axtell; Badawi; Coykendall; Frazier; Lauve; Lauveni; Levy; Livingston; Lucas; Maimani; Mulay; Naseer; Stickles; Smith; Wang; Wu; Yassemi ...

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Also, the zero divisor graph is used by R. Levy and J. Shapiro to characterize when $T(R)$ is von Neumann regular. In fact, they used for that the notion of complemented graph defined as follows:

Definition

Let x and y distinct vertices of $\Gamma(R)$.

- We say that x and y are orthogonal, written $x \perp y$, if x and y are adjacent and there is no vertex z of $\Gamma(R)$ which is adjacent to both x and y, i.e., the edge $x - y$ is not a part of any triangle of $\Gamma(R)$.
- We say that $\Gamma(R)$ is complemented if for each vertex x of $\Gamma(R)$, there is a vertex y of $\Gamma(R)$ such that $x \perp y$.

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Theorem (2002, Anderson, D.F., Levy, R., and Shapiro)

The following statements are equivalent for a reduced commutative ring R.

- *$T(R)$ is von Neumann regular.*
- *$\Gamma(R)$ is complemented.*

For that the Key Lemma was the following result.

Lemma

Let R be a ring and $a, b \in Z(R)^*$. Then the following statements are equivalent:

- *$a \perp b$, $a^2 \neq 0$ and $b^2 \neq 0$.*
- *$ab = 0$ and $a + b$ is a regular element of R.*

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Inspired by this result and a paper of D.F. Anderson and A. Badawi (2002), we have interested in the following zero divisor graph wich is used to give a sufficient condition so that the total quotient ring is zero dimentional.

Definition

Denoted by $\overline{\Gamma}(R)$ the simple graph associated to R with for distinct $x, y \in Z(R)^*$ the vertices x and y are adjacent if and only if there exist two non negative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$.

- The classical graph $\Gamma(R)$ is a partial graph of $\overline{\Gamma}(R)$.

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Distinct vertices x and y of $\overline{\Gamma}$ are orthogonal, written $x \perp_{\overline{\Gamma}} y$, if x and y are adjacent and there is no vertex z of $\overline{\Gamma}$ which is adjacent to both x and y, i.e., the edge $x - y$ is not a part of any triangle of $\overline{\Gamma}$. We say that $\overline{\Gamma}$ is complemented if for each vertex x of $\overline{\Gamma}$, there is a vertex y of $\overline{\Gamma}$ (called a complement of x) such that $x \perp_{\overline{\Gamma}} y$.

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Proposition

Let R be a ring with $\overline{\Gamma}(R) \neq \Gamma(R)$. If $\overline{\Gamma}(R)$ is complemented, then $T(R)$ is zero-dimensional.

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Let R be a ring and $x, y \in Z(R)^*$. If $x \perp_{\overline{\Gamma}} y$ with $x^2 \neq 0$ and $y^2 \neq 0$, then there are $n, m \in \mathbb{N}^*$ such that $x^n y^m = 0$ with $x^n \neq 0$, $y^m \neq 0$ and $x^n + y^m$ is a regular element of R.

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When $\overline{\Gamma}(R)$ and $\Gamma(R)$ coincide?

Theorem

Let R be a ring. The following statements are equivalent:

1. $\overline{\Gamma}(R) = \Gamma(R)$.
2. *R satisfies the two following conditions:*
   - (i) *If $Nil(R) \neq \{0\}$, then every nilpotent element has index 2, and*
   - (ii) *For all $x \in Z(R) \setminus Nil(R)$, $Ann(x^2) = Ann(x)$.*

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Let $(R_i)_{l \leq i \leq n}$ be a finite family of rings with $n \in \mathbb{N} \setminus \{0, 1\}$. Then $\overline{\Gamma}(\prod_{i=1}^{n} R_i) = \Gamma(\prod_{i=1}^{n} R_i)$ if and only if R_i is reduced for all i.

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As a simple consequence of Proposition, we determine when the graph $\overline{\Gamma}(\mathbb{Z}_n)$ coincides with $\Gamma(\mathbb{Z}_n)$.

Corollary

Let $n = \prod_{i=1}^{k} P_i^{\alpha_i}$ be the prime factorization of an integer n with $k \in \mathbb{N}^*$. Consider $m := \sup\{\alpha_i \mid 1 \leq i \leq k\}$. Then $\overline{\Gamma}(\mathbb{Z}_n) \neq \Gamma(\mathbb{Z}_n)$ if and only if either $m \geq 3$ or ($m = 2$ and $k \geq 2$).

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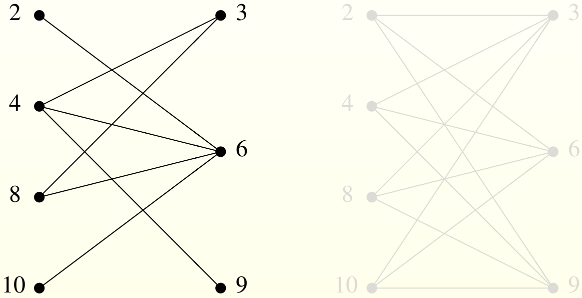

$\Gamma(\mathbb{Z}_{12})$ $\overline{\Gamma}(\mathbb{Z}_{12})$

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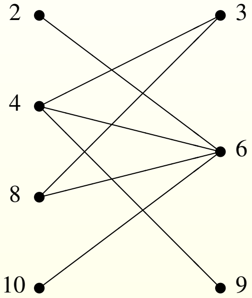

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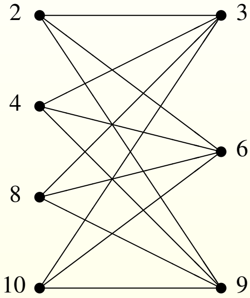

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In the following result $Ann_M(a)$, where $a \in R$, denotes the set of all elements of M annihilated by a. Also we use $Ann(M)$ to denote the annihilator of the R-module M.

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Let R be a commutative ring and let M be an R-module. Then $\overline{\Gamma}(R \ltimes M) = \Gamma(R \ltimes M)$ if and only if the following conditions hold true:

1. $(2Nil(R))M = 0$.
2. $\overline{\Gamma}(R) = \Gamma(R)$.
3. $\bigcup_{a \in \Lambda} Ann(a) \subset Ann(M)$ where $\Lambda = Z(R) \setminus Nil(R)$.
4. $Ann_M(a^2) = Ann_M(a)$ for all $a \in R \setminus Nil(R)$.

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Let R be a commutative ring and let M be an R-module. Then $\overline{\Gamma}(R \ltimes M) = \Gamma(R \ltimes M)$ if and only if the following conditions hold true:

1. $(2Nil(R))M = 0$.
2. $\overline{\Gamma}(R) = \Gamma(R)$.
3. $\bigcup_{a \in \Lambda} Ann(a) \subset Ann(M)$ *where* $\Lambda = Z(R) \backslash Nil(R)$.
4. $Ann_M(a^2) = Ann_M(a)$ *for all* $a \in R \backslash Nil(R)$.

WHEN $\overline{\Gamma}(R)$ AND $\Gamma(R)$ COINCIDE?

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Let R be a reduced ring and let M be an R-module. Then $\overline{\Gamma}(R \ltimes M) = \Gamma(R \ltimes M)$ if and only if the following conditions hold true:

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Let R be a ring such that $Z(R)$ is an ideal of R and let M be an R-module. Then $\overline{\Gamma}(R \ltimes R/Z(R)) = \Gamma(R \ltimes R/Z(R))$ if and only if $\overline{\Gamma}(R) = \Gamma(R)$.

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Let R be a ring. Then $\overline{\Gamma}(R \ltimes R/Nil(R)) = \Gamma(R \ltimes R/Nil(R))$ if and only if the following conditions hold true:

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Diameter of extended graphs of rings

Theorem

Let R be a ring. Then $\overline{\Gamma}(R)$ is connected with $diam(\overline{\Gamma}(R)) \leq 3$.

Diameter of extended graphs of rings

Theorem (1998, Anderson and Livingston)

Let R be a ring. Then, there is a vertex x of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain, or $Z(R) = Ann(x)$.

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Theorem

Let R be a ring. Then $\overline{\Gamma}(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R) = Nil(R)$ and for all $x, y \in Z(R)^*$ $x^{n_x - 1} y^{n_y - 1} = 0$.

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Proposition

Let $R = \prod_{i=1}^{n} R_i$ *where* $(R_i)_{l \leq i \leq n}$ *is a finite family of rings with* $n \in \mathbb{N} \backslash \{0, 1\}$.

1. *If* $n = 2$, *we have the following assertions:*
   1. $diam(\Gamma(R)) = diam(\overline{\Gamma}(R)) = 1$ *if and only if* $R_1 \cong R_2 \cong \mathbb{Z}_2$.
   2. *If* R_1 *and* R_2 *are integral with* $|R_1| \geq 3$ *or* $|R_2| \geq 3$, *then* $\Gamma(R) = \overline{\Gamma}(R)$ *and* $diam(\Gamma(R)) = 2$. *In this case* $\Gamma(R)$ *is a complete bipartite graph.*
   3. *If at least one of* R_1 *and* R_2 *contains a non-nilpotent zero divisor, then* $diam(\Gamma(R)) = diam(\overline{\Gamma}(R)) = 3$.
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2. *If* $n \geq 3$, $diam(\Gamma(R)) = diam(\overline{\Gamma}(R)) = 3$.

DIAMETER OF EXTENDED GRAPHS OF RINGS

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Cycles in extended graphs of rings

Theorem

Let R be a ring. If $\overline{\Gamma}(R) \neq \Gamma(R)$, then $\overline{\Gamma}(R)$ contains a cycle.

CYCLES IN EXTENDED GRAPHS OF RINGS

COROLLARY

If R contains a nilpotent element of index greater than or equal to three, then $gr(\overline{\Gamma}(R)) = 3$.

Thank you