How do you measure primality?

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Joint with Thomas Barron and Roberto Pelayo

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ω-primality

Definition (ω-primality)

Fix a cancellative, commutative, atomic monoid M. For $x \in M$, $\omega(x)$ is the smallest positive integer m such that whenever $x \mid \prod_{i=1}^{r} u_i$ for $r > m$, there exists a subset $T \subset \{1, \ldots, r\}$ with $|T| \le m$ such that $x \mid \prod_{i \in T} u_i$.

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Fact

$\omega(x) = 1$ if and only if x is prime (i.e. $x \mid ab$ implies $x \mid a$ or $x \mid b$).

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M is factorial if and only if every irreducible element $u \in M$ is prime. Moreover, $\omega(p_1 \cdots p_r) = r$ for any primes $p_1, \ldots, p_r \in M$.

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Definition

A *bullet* for $x \in M$ is a product $u_1 \cdots u_r$ of irreducible elements such that (i) x divides $u_1 \cdots u_r$, and (ii) x does not divide $u_1 \cdots u_r / u_i$ for each $i \leq r$. The set of bullets of x is denoted $\mathrm{bul}(x)$.

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Proposition

$\omega_M(x) = \max\{r : u_1 \cdots u_r \in \text{bul}(x)\}$.

Numerical monoids

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$n \in McN$	$\omega(n)$	bullet
6	3	$3 \cdot 20$
9	3	$3 \cdot 20$
12	3	$3 \cdot 20$
15	4	$4 \cdot 6$
18	3	$3 \cdot 6$

$n \in McN$	$\omega(n)$	bullet
20	10	$10 \cdot 6$
21	5	$5 \cdot 6$
24	4	$4 \cdot 6$
26	11	$11 \cdot 6$
27	6	$6 \cdot 6$

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Remark

Several improvements on this algorithm exist.

Quasilinearity for numerical monoids

Theorem ((O.–Pelayo, 2013), (García-García et.al., 2013))

$\omega_S(n) = \frac{1}{g_1}n + a_0(n)$ *for* $n \gg 0$, *where* $a_0(n)$ *periodic with period* g_1.

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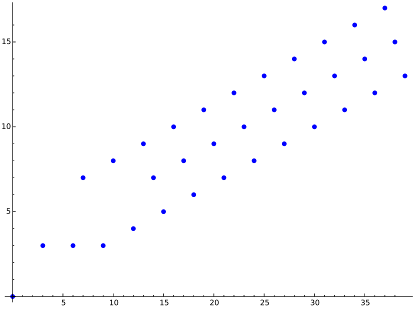
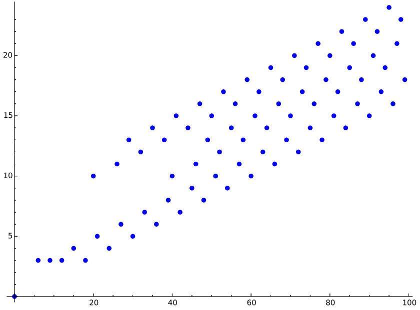


$S = \langle 3, 7 \rangle$



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Answer (Barron-O.-Pelayo, 2014)

Yes!

Toward a dynamic algorithm. . . the inductive step

For $n \in S$, let $\mathsf{Z}(n) = \{\vec{a} \in \mathbb{N}^k : \sum_{i=1}^k a_i g_i = n\}$.

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Fix $n \in S$ and $i \leq k$. The *i-th cover morphism for n* is the map

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Moreover, $\mathrm{bul}(n) = \bigcup_{i \leq k} \psi_i(\mathrm{bul}(n - g_i))$.**

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Proposition

For $n \in \mathbb{Z}$, the following are equivalent:

(i) $\omega(n) = 0$,

(ii) $\mathsf{bul}(n) = \{\vec{0}\}$,

(iii) $-n \in S$.

A dynamic algorithm!

Example

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$\vdots$	$\vdots$	$\vdots$

$n \in \mathbb{Z}$	$\omega(n)$	$\text{bul}(n)$
1	5	$\{5\vec{e}_1, (2, 1, 0), \ldots\}$

A dynamic algorithm!

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≤ -44	0	$\{\vec{0}\}$
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$\vdots$	$\vdots$	$\vdots$
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9	3	$\{3\vec{e}_1, 3\vec{e}_3, \ldots\}$
$\vdots$	$\vdots$	$\vdots$

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$\vdots$	$\vdots$	$\vdots$
148	28	$\{28\vec{e}_1, \ldots\}$

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Runtime comparison

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S	$n \in S$	$\omega_S(n)$	Existing	Dynamic
$\langle 6, 9, 20 \rangle$	1000	170	0m 67s	0.4s
$\langle 6, 9, 20 \rangle$	2000	340	17m 20s	3.1s
$\langle 31, 39, 45, 52 \rangle$	1000	40	0m 39s	0.3s
$\langle 31, 39, 45, 52 \rangle$	2000	71	24m 31s	2.0s
$\langle 54, 67, 69, 73, 75 \rangle$	1000	23	2m 02s	0.7s
$\langle 54, 67, 69, 73, 75 \rangle$	1500	33	22m 50s	2.3s
$\langle 54, 67, 69, 73, 75 \rangle$	3000	61	———	44.3s
$\langle 54, 67, 69, 73, 75 \rangle$	5000	98	———	8m 59.0s

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Sage: Open Source Mathematics Software, available at

`www.sagemath.org`.

GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

Future directions

What about more general (finitely generated) monoids M?

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- Characterization of ω_M in terms of maximal length bullets?

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Problem:

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Problem: the base case!

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Problem: the base case!

Problem

Find a dynamic algorithm to compute ω-primality in M.

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Question

Are there dynamic algorithms for computing other factorization invariants?

References

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Thanks!