Noetherian semigroup algebras and beyond

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Plan of the talk

1. Two motivating introductory results

   i) Noetherian group algebras
   ii) commutative semigroup algebras

2. Finitely generated algebras

3. A search for necessary and sufficient conditions

   i) structural necessary/sufficient conditions
   ii) submonoids of polycyclic-by-finite groups

4. Algebras with homogeneous relations

5. Important motivating examples

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Notation

K will denote a field

S - a semigroup (in most cases a monoid)

$K[S]$ - the corresponding semigroup algebra

if S has a zero element θ then $K_0[S] = K[S]/K\theta$ is called the contracted semigroup algebra
(in other words: identify the zero of S with the zero of the algebra)

Two motivating classical results

Theorem (folklore)

Let G be a polycyclic-by-finite group. Then $K[G]$ is Noetherian.

Idea of the proof (easy): induction on the length of a subnormal chain of G with finite and cyclic factors. Let $H \subseteq F$ be consecutive factors of such a chain. Assume that $K[H]$ is Noetherian.
- If $[F : H] < \infty$, then we have a finite module extension $K[H] \subseteq K[F]$. So $K[F]$ is Noetherian.
- If F/H is infinite cyclic, then an argument similar to that in the proof of Hilbert basis theorem is used to show that $K[F]$ is also Noetherian.

Note: it is not known whether there are another classes of examples of Noetherian group algebras!

Theorem (Budach, 1964)

Assume that S is a commutative monoid. If $K[S]$ is Noetherian then S is finitely generated.

The proof is based on a decomposition theory for congruences of a commutative monoid with acc on congruences, on properties of irreducible congruences and of cancellative congruences, see R.Gilmer, Commutative Semigroup Rings.

Finitely generated algebras

Theorem (JO)

Assume $K[S]$ is right Noetherian. Then S is finitely generated in each of the following cases:

1. *S satisfies acc on left ideals (this holds in particular if $K[S]$ is also left Noetherian),*
2. *$K[S]$ satisfies a polynomial identity,*
3. *the Gelfand-Kirillov dimension of $K[S]$ is finite.*

Problem: Is the assertion true for arbitrary right Noetherian $K[S]$?

This is not known even in the case where S is cancellative (so it has a group of (classical) quotients, because of the acc on right ideals).

Idea of the proof:

Consider the image $\overline{S}$ of S under the natural map $K[S] \longrightarrow K[S]/B(K[S])$, where $B(K[S])$ denotes the prime radical of $K[S]$. Then:

i) show that $\overline{S}$ is finitely generated by exploiting the structure of $\overline{S}$ as a subsemigroup of the matrix algebra $M_n(D)$ over a division algebra D,

ii) lift this condition to S by using acc on right ideals in S (not difficult).

Structural approach

Let X, Y be arbitrary nonempty sets and let P be a $Y \times X$-matrix with entries in $G^0 = G \cup \{0\}$. Assume that P has no nonzero rows or columns. Let $M(G, X, Y, P)$ be the set of all $X \times Y$-matrices with at most one nonzero entry in G. Such a nonzero matrix can be denoted by (g, x, y) (with g in position (x, y)). Multiplication is defined as follows:

$$g \circ h = gPy.$$

Then $M(G, X, Y, P)$ is called a completely 0-simple semigroup over a group G with sandwich matrix P.

A subsemigroup S of $M(G, X, Y, P)$ such that S intersects nontrivially every set $M_{xy} = \{(g, x, y) \mid g \in G\}$ is called a uniform semigroup.

A special case is when $X = Y$ and $P = \Delta$, the identity matrix. If $|X| = r$ then we write $M(G, r, r, \Delta)$.

Note that the contracted semigroup algebra $K_0[M(G, r, r, \Delta)]$ is isomorphic to the matrix algebra $M_r(K[G])$. A uniform subsemigroup S of $M(G, r, r, \Delta)$ is called a semigroup of generalized matrix type.
So $K_0[S] \subseteq M_r(K[G])$.

Let $S \subseteq M(G, X, Y, P)$ be a uniform semigroup.
One can show that there exists a unique subgroup H of G, and a sandwich matrix Q over H, such that one can consider $M(H, X, Y, Q)$ as a "semigroup of quotients of S". If one prefers, one can consider S as an "order" in $M(H, X, Y, P)$.

Note that $M(G, X, Y, P)$ plays in semigroup theory the role played in ring theory by a simple artinian ring.

Now, if S is a subsemigroup of the multiplicative monoid $M_n(F)$ of all $n \times n$-matrices over a field F, then S has an ideal chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k = S$ with I_1 and every factor I_j/I_{j-1} nilpotent or a uniform semigroup.
For example, if $S = M_n(F)$, then the chain $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M_n(F)$ has all factors completely 0-simple; where $M_j = \{a \in M_n(F) \mid rank(a) \leq j\}$.

Why is the former relevant?

Theorem (Ananin)

Let R be a a finitely generated right Noetherian PI-algebra. Then R embeds into the matrix ring $M_n(F)$ over a field extension F of the base field K.

Important classes of semigroup algebras which fit in this cotext:

Theorem (Gateva-Ivanova, Jespers, JO)

Assume that $K[S]$ is right Noetherian and $\mathrm{GKdim}(K[S]) < \infty$. If S has a presentation of the form $S = \langle x_1, \ldots, x_n \mid R \rangle$ where R is a set of homogeneous (semigroup) relations, then $K[S]$ satisfies a polynomial identity.

An important consequence: S has a finite ideal chain in S with factors nilpotent or uniform!
(Because: 1) S is finitely generated by a previous theorem and 2) every multiplicative semigroup of matrices has such a chain.)

Theorem (JO)

Let S be a finitely generated monoid with an ideal chain $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n = S$ such that S_1 and every factor S_i/S_{i-1} is either nilpotent or a semigroup of generalized matrix type. If $GKdim(K[S]) < \infty$ and S satisfies the ascending chain condition on right ideals then $K[S]$ is right Noetherian.

The proof shows that that cancellative subsemigroups of uniform factors S_i/S_{i-1} and S_1 have groups of quotients that are finitely generated and nilpotent-by-finite (so polycyclic-by-finite).

Submonoids of polycyclic-by-finite groups

Theorem (Jespers, JO)

Let S be a submonoid of a polycyclic-by-finite group G. Then the following conditions are equivalent:

1. $K[S]$ *is right Noetherian,*
2. S *satisfies acc on right ideals,*
3. *there exists a normal subgroup H of G such that:* $[G : H] < \infty$, $S \cap H$ *is finitely generated and* $[H, H] \subseteq S$,
4. $K[S]$ *is left Noetherian.*

Let $F = [H, H]$. So, in some sense, such $K[S]$ can be approached from the perspective of the Noetherian group algebra $K[F] \subseteq K[S]$ and the Noetherian PI-algebra $K[S/F] \subseteq K[G/F]$.

It follows, in this case, that S and $K[S]$ are finitely presented.

Important motivating examples

Important classes of examples include algebras corresponding to set theoretic solutions of the Yang-Baxter equation.
By a set theoretic solution of the Yang-Baxter equation we mean a map $r : X \times X \to X \times X$, where $X = \{x_1, \dots, x_n\}$ is a set, such that

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where r_{ij} denotes the map $X \times X \times X \to X \times X \times X$ acting as r on the (i, j) factor and as the identity on the remaining factor.

One considers solutions that are involutive ($r^2 = id$) and non-degenerate (will be defined later).
One associates to r an algebra defined by the presentation $K\langle x_1, \dots, x_n\rangle/J$ where J consists of relations of the form $xy = x'y'$ if $r(x, y) = (x', y')$.
This implies that J consists of $\binom{n}{2}$ relations.

Theorem (Gateva-Ivanova, Van den Bergh)

These algebras are isomorphic to $K[S]$, where S is a submonoid of a finitely generated torsion free abelian-by-finite group. They are Noetherian PI domains of finite homological dimension and they are maximal orders.

Simplest examples.

1. $S =$ free commutative monoid,
2. $S = \langle x, y \mid x^2 = y^2 \rangle$.

These algebras have several other properties similar to the properties of commutative polynomial rings. New examples are very difficult to construct.

Height one prime ideals P are of a very special form:

$$P = aK[S] = K[S]a$$

for some $a \in S$ and there are finitely many height one primes. In particular, this can be used to prove that $K[S]$ is a maximal order.

More general: quadratic monoids of skew type.

These are monoids with generators $x_1, x_2, \ldots, x_n$ subject to $\binom{n}{2}$ quadratic relations of the form $x_i x_j = x_k x_l$ with $(i,j) \neq (k,l)$ and, moreover, every monomial $x_i x_j$ appears at most once in one of the defining relations.

For every $x \in X = \{x_1, \ldots, x_n\}$, let

$$f_x : X \to X$$

and

$$g_x : X \to X$$

be the maps such that

$$r(x,y) = (f_x(y), g_y(x)).$$

One says that S is non-degenerate if each f_x and each g_x is bijective, with $x \in X$.

Theorem (Jespers, JO, Van Campenhout and Cedo, JO)

Let S be a non-degenerate monoid of skew type. Then $K[S]$ is right and left Noetherian and it is a finitely generated module over a commutative subalgebra of the form $K[A]$, for a submonoid A of S.

The proof uses the structural approach explained before:
- first, certain ideal chain in S is constructed from the combinatorial data, in order to prove $K[S]$ is Noetherian,
- also, one shows that $GKdim(K[S]) < \infty$,
- then we get PI by the theorem on semigroups defined by homogeneous relation,
- finally, using the embedding theorem of Anan'in, we get that S is a semigroup of matrices, and using this and with more work we get the last assertion.

Maximal orders

A monoid S which has a left and right group of fractions G is called an order.
Then S is called a maximal order if there does not exist a submonoid S' of G properly containing S and such that $aS'b \subseteq S$ for some $a, b \in G$.
A nonempty subset I of G is called a fractional ideal of S if $SIS \subseteq I$ and $cI, Id \subseteq S$ for some $c, d \in S$.

Assume now that S is a maximal order. For subsets $A, B \subseteq G$ let $(A :_l B) = \{g \in G \mid gB \subseteq A\}$ and $(A :_r B) = \{g \in G \mid Bg \subseteq A\}$.
Then $(S :_r I) = (S :_l I)$ for any fractional ideal I. One denotes this set as $(S : I)$. Put $I^* = (S : (S : I))$, the divisorial closure of I. If $I = I^*$ then I is said to be divisorial.
S is said to be a Krull order if S satisfies also the ascending chain condition on divisorial ideals contained in S. In this case the divisor group $D(S)$ (also defined as in ring theory) is a free abelian group.

Theorem (Chouinard)

A commutative monoid algebra $K[S]$ is a Krull domain if and only if S is a submonoid of a torsion-free abelian group which satisfies the ascending chain condition on cyclic subgroups and S is a Krull order in its group of quotients.
Furthermore, S is a Krull monoid if and only if S is the direct product of its unit group $U(S)$ and a monoid $A = AA^{-1} \cap F^{+}$, where F_{+} is the positive cone of a free abelian monoid F.
Moreover, in this situation the class group of $K[S]$ equals the class group of S.

The last property mentioned in the theorem allows one to simplify the calculation of the class group of in several concrete classes of examples, and it also shows that the height one primes of $K[S]$ determined by the minimal primes of S are crucial.

K.Brown characterized group algebras $K[G]$ of polycyclic-by-finite groups G that are prime Noetherian maximal orders. In the PI-case this turns out to be always the case if the group is also torsion-free.

Theorem (Brown)

Let G be a finitely generated torsion-free abelian-by-finite group. Then the group algebra $K[G]$ is a Noetherian maximal order. Moreover, all height one primes of $K[G]$ are principally generated by a normal element.

An extension to the polycyclic-by-finite case is also known. Let G be a polycyclic-by-finite group. Then, $K[G]$ is a prime Noetherian maximal order if and only if $\Delta^{+}(G) = \{1\}$ (this means that G has no nontrivial finite normal subgroups) and G is dihedral free.

G dihedral free: if $H \subseteq G$ and $H \cong D_\infty$ then H has infinitely many conjugates.

If S is a monoid with a torsion-free abelian-by-finite group of quotients G (so $K[S]$ is a PI-domain), the maximal order property of $K[S]$ is determined by the structure of S and can be reduced to some "local" monoids S_P, with P a minimal prime ideal of S. Here

$$S_P = \{g \in G \mid Cg \subseteq S \text{ for some } G\text{-conjugacy class } C \text{ of } G \text{ contained in } S \text{ with } C \not\subseteq P\}.$$

Theorem (Goffa, Jespers, JO)

Let be a submonoid of a finitely generated torsion-free abelian-by-finite group. Then the monoid algebra is a Noetherian maximal order if and only if the following conditions are satisfied:

1. *S satisfies the ascending chain condition on one-sided ideals,*
2. *S is a maximal order in its group of quotients,*
3. *for every minimal prime ideal P of S the monoid S_P has only one minimal prime ideal.*

Furthermore, in this case, each S_P is a maximal order satisfying the ascending chain condition on one-sided ideals.

Recent results - a non-Noetherian example

Let

$$G = gr(a, b, c \mid ac = ca, ab = ba, bc = acb)$$

the Heisenberg group (a nilpotent group of class 2).
Let

$$M = \langle x, y, z, t \mid xy = yx, zt = tz, yz = xt = zx, zy = tx = yt \rangle$$

a finitely presented monoid. So $K[M]$ carries some similarity to Y-B algebras ($\binom{n}{2}$ quadratic relations).
It can be shown that: $\phi : M \longrightarrow G$ defined by

$$x \mapsto c, y \mapsto ac, z \mapsto bc, t \mapsto abc$$

is a homomorphism which also is an embedding. Hence

$$M \cong \phi(M) \subseteq G.$$

Note: $K[M]$ is an Ore domain, but it is not Noetherian (use one of theorems above: G is not abelian-by-finite while M has trivial units; but this is also not hard to check directly).

$K[M]$ is the algebra used by Yekutieli and Zhang (as a counterexample in the context of AS-regular rings), and recently by Rogalski and Sierra - where it plays a key role in the classification of 4-dimensional non-commutative projective surfaces.

Namely, a family of deformations of $K[M]$ is considered. They are of the form:

$$R(\rho, \theta) = K\langle x_1, x_2, x_3, x_4 \mid f_i = 0, i = 1, 2, 3, 4, 5, 6\rangle$$

where

$$\begin{aligned}
f_1 &= x_1(cx_1 - x_3) + x_3(x_1 - cx_3) \\
f_2 &= x_1(cx_2 - x_4) + x_3(x_2 - cx_4) \\
f_3 &= x_2(cx_1 - x3) + x_4(x_1 - cx_3) \\
f_4 &= x_2(cx_2 - x_4) + x_4(x_2 - cx_4) \\
f_5 &= x_1(dx_1 - x_2) + x_4(x_1 - dx_2) \\
f_6 &= x_1(dx_3 - x_4) + x_4(x_3 - dx_4)
\end{aligned}$$

for $c = (\theta - 1)/(\theta + 1)$ and $d = (\rho - 1)(\rho + 1)$.

Notice that $R(1,1) \cong K[M]$ and it is embeddable in the skew polynomial ring $K(u,v)[t,\sigma]$ over the rational function field $K(u,v)$, where $\sigma(v) = v, \sigma(u) = uv$.

Theorem (Rogalski, Sierra)

If ρ, θ are algebraically independent over the prime subfield of K then $R(\rho,\theta)$ is a Noetherian domain of global dimension 4 and GK-dimension 4. And it is birational to $\mathbb{P}^2$.

(For a connected graded Noetherian domain we have $Q_{gr}(R) \cong D[t, t^{-1}, \sigma]$, and if the division ring D is a field (then $D = K(X)$ for a projective variety X), then R is said to be birational to X.)

So, this is a new motivation to study algebras of submonoids of nilpotent groups that are not necessarily Noetherian.

Theorem (Jespers, JO)

Let S be a submonoid of a finitely generated torsion-free nilpotent group. Then the following properties hold.

1. *S is a maximal order if and only if $K[S]$ is a maximal order.*
2. *If, moreover, S satisfies the ascending chain condition on right ideals and is a maximal order then all elements of S are normal (meaning: $aS = Sa$ for every $a \in S$).*

So, in the latter case, the theorem below applies.

Theorem (Jespers, JO)

Let S be a submonoid of a torsion-free polycyclic-by-finite group. Assume that all elements of S are normal. Then the following conditions are equivalent:

1. *$K[S]$ is a Krull domain,*
2. *S is a Krull order,*
3. *$S/U(S)$ is an abelian Krull order.*

Theorem (Jespers, JO)

Assume that S is a submonoid of a finitely generated nilpotent group of class two. Assume that $K[S]$ is a Krull order. Then

(i) *the derived subgroup G' of the quotient group G of S is contained in S,*

(ii) $S = N(S)$,

(iii) *S/G' is a commutative Krull order,*

(v) *$K[S]$ is a Krull domain for every field K.*

On the other hand, if $G' \subseteq S$ and S/G' is a Krull order then S is a Krull order. If, furthermore, G is finitely generated then S is finitely generated and $K[S]$ is right and left Noetherian.

Problem: what about nilpotent groups of a higher nilpotency class?

Some references

F. Cedo, J. Okniński, Gröbner bases for quadratic algebras of skew type, Proc. Edinburgh Math. Soc. 55 (2012), 387-401.

R. Gilmer, Commutative Semigroup Rings, Univ. Chicago Press, Chicago, 1984.

E. Jespers, J. Okniński, Noetherian semigroup algebras, Algebras and Applications 7, Springer, Dordrecht, 2007.

E. Jespers, J. Okniński, Krull orders in nilpotent groups, Arch. Math. 103 (2014), 27-37.

D. Rogalski, S.J. Sierra, Some noncommutative projective surfaces of GK-dimension 4, Compos. Math. 148 (2012), 1195–1237.