Ore and Ore-Rees rings which are maximal orders

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$R$: Noetherian prime ring with quotient ring $Q$

$\sigma$: an automorphism of $R$ and $\delta$: a left $\sigma$-derivation, that is

(i) $\delta(a + b) = \delta(a) + \delta(b)$ and
(ii) $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$.

Ore extension of $R$ is:

$R[t; \sigma, \delta] = \{ f(t) = a_n t^n + \cdots + a_0 | a_i \in R \}$, where $t$ is an indeterminate and the multiplication is defined by:

$ta = \sigma(a)t + \delta(a)$, ($a \in R$).

$X$: invertible ideal of $R$, that is

$XX^{-1} = R = X^{-1}X$

$S = R[Xt; \sigma, \delta] = R \oplus Xt \oplus \cdots \oplus X^n t^n \oplus \ldots$

which is a subset of $R[t; \sigma, \delta]$

It is easy to see that $S$ is a ring if and only if $\sigma(X) = X$.

$S$ is called an Ore-Rees ring if $S$ is a ring.

1 General Theory of Ore-Rees Rings

Notation: For any $R$-ideal $a$, $(R : a)_l = \{ q \in Q | qa \subseteq R \}$, $(R : a)_r = \{ q \in Q | aq \subseteq R \}$.

$a_v = (R : (R : a)_l)_r \supseteq a$. and $a = (R : (R : a)_r)_l \supseteq a$.

If $\vDash a = a$ ($a = a_v$), then $a$ is called a left divisorial (right divisorial) ideal.

If $\vDash a = a = a_v$, then $a$ is just called a divisorial $R$-ideal.

Definition $R$ is a maximal order if $O_l(a) = R = O_r(a)$ for all ideals $a$ of $R$, where $O_l(a) = \{ q \in Q | qa \subseteq a \}$, the left order of $a$ and $O_r(a) = \{ q \in Q | aq \subseteq a \}$, the right order of $a$.

$\Leftrightarrow$ Any divisorial $R$-ideal $a$ is divisorially invertible, that is $(aa^{-1})_v = R = v(a^{-1}a)$

$D(R) = \{ a: divisorial R-ideal in Q \}$
is an Abelian group under the multiplication \( \circ \); \( a \circ b = (ab)_v \).

**Examples of Maximal orders**

(1) In case of commutative domains, maximal orders \( \iff \) completely integrally closed domains.

(2) In case of non-commutative rings,

(a) Algebra case:

\[
\begin{array}{ccc}
Q & \rightarrow & R \\
K & \rightarrow & R \\
D & \rightarrow & D \\
\end{array}
\]

Let \( Q \) be a simple Artinian ring with \( [Q : K] < \infty \), where \( K = \mathbb{Z}(Q) \) is the center of \( Q \). A subring \( R \) of \( Q \) with the center \( D \) is called a **D-order** in \( Q \) if the following are satisfied:

(i) \( K = Q(D) \), the quotient field of \( D \) and \( KR = Q \).

(ii) Every element of \( R \) is integral over \( D \).

Then:

(1) There always exists a maximal \( D \)-order in \( Q \).

(2) If \( D \) is a Dedekind domain, then any maximal order is a non-commutative Dedekind ring (see, I. Reiner: Maximal orders, Academic Press, 1975).

(3) If \( D \) is a Krull domain, then any maximal order is a non-commutative Krull ring (see, e.g. H. Marubayashi and F. Van Oysteayen: Prime Divisors and Noncommutative Valuation Theory, Lecture Notes in Math. 2059, Springer, 2012).

(b) If \( R \) is a maximal order, then the Ore-extension \( R[t; \sigma, \delta] \) and skew formal power series ring \( R[[t; \sigma]] \) are maximal orders.

(c) Non-commutative Krull rings, unique factorization rings and regular rings are all maximal orders.

(e) The class of maximal orders are occupied the important parts in enveloping algebras, crossed product algebras (including group rings) and semi-group algebras (E. Jespers and J. Okininski’s book: Noetherian Smigroup Algebras).
• \((\sigma, \delta)\) are naturally extended to the automorphism \(\sigma\) of \(Q\) and the left \(\sigma\)-derivation on \(Q\) by \(\sigma(ac^{-1}) = \sigma(a)\sigma(c)^{-1}\) and \(\delta(c^{-1}) = -\sigma(c^{-1})\delta(c)c^{-1}\) for any \(a, c \in R\) and \(c\) is a regular element in \(R\).

Let \(T = Q[t; \sigma, \delta]\), Ore extension of \(Q\). It is well known that
(i) \(T\) is a principal ideal ring and
(ii) \(AT\) is an ideal of \(T\) for each ideal \(A\) of \(S\) (the proof is not difficult).

**Theorem 1.1.** If \(R\) is a maximal order, then \(S = R[Xt; \sigma, \delta]\) is a maximal order. But the converse is not true.

The outline of the proof:

Let \(A\) be an ideal of \(S\). We need to prove: \(O_l(A) = S\) (it is clear that \(O_l(A) \supseteq S\)). To show the converse inclusion, Let \(q \in O_l(A)\). We have:
(i) \(q \in T\), that is \(q = q_1t^n + \cdots + q_0 (q_i \in Q)\)
(ii) \(C_m(A) = \{ a \in R | \exists h(t) = at^n + \cdots + a_0 \in A \} \cup \{0\}\). Then:

\[ X^k\sigma^k(C_m(A)) = C_{m+k}(A) \]

for some \(m\) and for any \(k\).

• In order to study the properties of divisorial ideals of \(S = R[Xt; \sigma, \delta]\), we need the following concept:

**Definition.** An \(R\)-ideal \(a\) in \(Q\) is called \((\sigma, \delta; X)\)-stable if \(X\sigma(a) = aX\) and \(X\delta(a) \subseteq a\).

In case \(\delta = 0\), \(a\) is called \((\sigma; X)\)-invariant.

**Lemma 1.2.** Let \(a\) be an ideal of \(R\). Then \(A = a[Xt; \sigma, \delta]\) is an ideal of \(S\) if and only if \(a\) is \((\sigma, \delta; X)\)-stable.

**Definition.** \(R\) is a \((\sigma, \delta; X)\)-maximal order in \(Q\) if \(O_l(a) = R = O_r(a)\) for any \((\sigma, \delta; X)\)-stable ideal \(a\).

**Proposition 1.3.** If \(R\) is a \((\sigma, \delta; X)\)-maximal order, then \(D(\sigma, \delta; X)(R) = \{ a : \sigma, \delta\)-stable and divisorial \(R\)-ideals \}\) is an Abelian group generated by maximal \((\sigma, \delta; X)\)-stable divisorial ideals.

**Proposition 1.4.** Suppose \(R\) is a \((\sigma, \delta; X)\)-maximal order. Let \(P\) be a prime ideal of \(S\) such that \(P \cap R = p\) is \((\sigma, \delta; X)\)-stable and \(P\) is divisorial. Then \(P = p[Xt; \sigma, \delta]\).
2 Differencial Rees ring

In case $\sigma = 1$ and $\delta \neq 0$, $S = R[\!\!\![X; 1, \delta]\!\!\!] = R[\!\!\![X; \delta]\!\!\!]$ is called a differencial Rees ring.

- An $R$-ideal $\mathfrak{a}$ in $Q$ is called $(\delta; X)$-stable if $X\mathfrak{a} = \mathfrak{a}X$ and $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$.
- $R$ is a $(\delta; X)$-maximal order if $O_r(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for all $(\delta; X)$-stable ideal $\mathfrak{a}$ of $R$.

**Theorem 2.1.** $S = R[\!\!\![X; \delta]\!\!\!]$ is a maximal order if and only if $R$ is a $(\delta; X)$-maximal order.

- The outline of the proof: If $R$ is a $(\delta; X)$-maximal order, then $S = R[\!\!\![X; \delta]\!\!\!]$ is a maximal order.

**Lemma 2.2.** Let $P$ be a prime ideal of $S$ with $P = P_v \ (P = v P)$. Then $\mathfrak{p} = P \cap R$ is $(\delta; X)$-stable.

This lemma is proved by considering two cases: either $P \supseteq X$ or $P \not\supseteq X$.

**Lemma 2.3** Suppose $R$ is a $(\delta; X)$-maximal order. Let $A$ be a divisorial ideal of $S$. If $\mathfrak{a} = A \cap R \neq (0)$. Then

1. $\mathfrak{a}$ is a $(\delta; X)$-stable divisorial ideal and
2. $A = \mathfrak{a}[\!\!\![X; \delta]\!\!\!]$. In particular, $A$ is divisorially invertible.

The outline of the proof.

Put $\mathfrak{B} = \{A: \text{ideal} \ | \ A \cap R \neq 0 \text{ and } A = A_v \}.$
If $P$ is maximal in $\mathfrak{B}$, then $P$ is a prime ideal and so $P = P[X; \delta]$ by Proposition 1.4 and Lemma 2.2.

**Lemma 2.4.** Suppose $R$ is a $(\delta; X)$-maximal order. Let $A$ be a divisorial ideal with $A \cap R = (0)$. Then $A$ is a divisorially invertible ideal.

This lemma is proved by the following way: $A(S : A)_v \cap R \neq (0)$ and, by Lemma 2.3, $(A(S : A)_v)_v = a[\!\!\![X; \delta]\!\!\!]$, where $a = (A(S : A)_v \cap R)_v.$

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\[ ((A(S : A)r)a^{-1}|Xt; \delta))v = S. \] Hence \( A \) is divisorially invertible.

**Theorem 2.5.** Suppose \( R \) is a \((\delta; X)\)-maximal order. Then any divisorial \( S \)-ideal \( A \) is of the form:

\[ A = wa[Xt; \delta] \]

where \( w \in \mathbb{Z}(Q(T)) \) = the center of \( Q(T) \) and \( a \) is a \((\delta; X)\)-stable divisorial \( R \)-ideal.

This is easily proved by Lemmas 2.3, 2.4 and the fact: any ideal \( A' \) of \( T \) is of the form: \( A' = wT \) for some \( w \in \mathbb{Z}(T) \) (see: G.Cauchon’s Ph.D thesis, 1977).

### 3 Skew Rees rings

In case \( \delta = 0 \), \( S = R[Xt; \sigma, 0] = R[Xt; \sigma] \) is called a **skew Rees ring**.

- An ideal \( a \) of \( R \) is **(\( \sigma; X \))-invariant** if
  \( X\sigma(a) = aX \)
- \( R \) is a **(\( \sigma; X \))-maximal order** if
  \( O_\sigma(a) = R = O_t(a) \) for all \((\sigma; X)\)-invariant ideal \( a \) of \( R \).

**Theorem 3.1.** \( S = R[Xt; \sigma] \) is a maximal order if and only if \( R \) is a \((\sigma; X)\)-maximal order.

**Theorem 3.2.** Suppose \( R \) is a \((\sigma; X)\)-maximal order. Then any divisorial \( S \)-ideal \( A \) is of the form:

\[ A = t^nwa[Xt; \sigma] \]

where \( w \in \mathbb{Z}(Q(T)) = \) the center of \( Q(T) \) and \( a \) is a divisorial \((\sigma; X)\)-invariant \( R \)-ideal, \( n \) is an integer and \( Q(T) \) is the quotient ring of \( T = Q[t; \sigma] \).

### 4 Unique factorization rings (UFRs) and Krull rings

**Definition.** (1) \( R \) is a **UFR** in the sense of Chatters and Jordan(C-J) if any non-zero prime ideal contains a principal prime ideal (see: Kaplansky’s book).

(2) \( R \) is a **UFR** in the sense of mine if for any prime ideal \( P \) with \( P = P_v \) \((P = vP) \) is principal, that is \( aR = P = Ra \) for some \( a \in R \) (see: Samuel’s Lecture from Tata Institute).

**Definition** \( R \) is called a **Krull ring** in the sense of Chamarie if

(1) \( R \) is a maximal order.

(2) \( R \) satisfies the a.c.c. on one sided "closed" ideals.
(1) \( R = \cap R_P \cap S(R) \), where \( R_P \) is a local Dedekind prime ring (\( P \) runs over all prime divisorial ideals of \( R \)) and \( S(R) = \cup A^{-1} \) is a divisorially simple (\( A \) runs over all ideals of \( R \))

(iii) For any regular element \( c \) in \( R \), \( cR_P = R_P \) for almost all \( P \).

\[ (\text{UFRs of C-J}) \implies (\text{Krull rings of mine}) \]
\[ (\text{UFRs of mine}) \implies (\text{Krull rings of Chamarie}) \]

(ii) If \( R \) is a UFR in the sense of mine, then so are \( R[t, \sigma] \) and \( R[t, \delta] \). But if \( R \) is a UFR in the sense of (C-J), then both \( R[t, \sigma] \) and \( R[t, \delta] \) are not necessary to be UFRs in the sense of (C-J).

5 Open questions and Remarks

Question 5.1. (Long standing open question) Find out a necessary and sufficient conditions for Ore extension \( R[t; \sigma, \delta] \) to be a maximal order (a UFR) and describe the structure of divisorial ideals (see: M.Chamarie, Anneaux de Krull non commutative, these, 1981).

Question 5.2. Find out a necessary and sufficient conditions for \( S = R[\pi t; \delta] \) to be a maximal order (a UFR) and describe the structure of divisorial ideals of \( S \).

- In case \( \delta = 0 \), the skew Rees ring \( R[\pi t; \sigma] \) is a UFR \iff \( X \) is principal and \( R \) is a \( (\pi; X) \)-UFR, that is any \( (\pi; X) \)-invariant divisorial ideal is principal.

- In case \( \sigma = 1 \), the differential Rees ring \( R[\pi t; \delta] \) is a UFR \iff \( X \) is principal and \( R \) is a \( (\delta; X) \)-UFR, that is any \( (\delta; X) \)-stable divisorial ideal is principal.

\[ \text{(Dedekind rings)} \rightarrow \text{(Asano rings)} \]
\[ \downarrow \]
\[ \text{(Hereditary rings)} \]

Definition. (1) \( R \) is Dedekind \iff it is a maximal order and any one sided ideal is projective.

(2) \( R \) is Asano if any ideal is invertible.

(3) \( R \) is hereditary if any one sided ideal is projective.
The following is the simplest examples such that these three concepts are
different:

\[ R = \left( \frac{\mathbb{Z}}{\mathbb{Z}}, \frac{\mathbb{Z}}{\mathbb{Z}}, \frac{\mathbb{Z}}{\mathbb{Z}} \right), \text{Dedekind} \]

\[ S = \left( \frac{\mathbb{Z}}{\mathbb{Z}}, \frac{p\mathbb{Z}}{\mathbb{Z}}, \frac{\mathbb{Z}}{\mathbb{Z}} \right), \text{hereditary and but neither Dedekind nor Asano} \]

where \( \mathbb{Z} \) is the ring of integers and \( p \) is a prime number.

\[ M = \left( \frac{p\mathbb{Z}}{\mathbb{Z}}, \frac{p\mathbb{Z}}{\mathbb{Z}}, \frac{\mathbb{Z}}{\mathbb{Z}} \right), \quad N = \left( \frac{\mathbb{Z}}{\mathbb{Z}}, \frac{p\mathbb{Z}}{\mathbb{Z}}, \frac{p\mathbb{Z}}{\mathbb{Z}} \right) \Rightarrow M = M^2 \]

and \( N = N^2 \).

Let \( R \) be a Noetherian simple ring but not Artinian (e.g. the nth- Weyl
algebra with \( \text{char } R = 0 \) for any \( n \)) and \( R[t] \), the polynomial over \( R \).
Then \( R[t] \) is Asano but neither Dedekind nor hereditary.

- Let \( R \) be a hereditary ring and \( S = R[t] \), the polynomial ring. Then \( S \) is
  not hereditary if \( R \neq \mathbb{Q} \).

- what kind of properties does \( R[t] \) have from the arithmetical point of view?
- Any ideal \( A \) of \( R[t] \) with \( A = A_v \) (or \( A = vA \)) is left and right projective.

This fact leads us to define the following:

**Definition.** \( R \) is called a **generalized hereditary** if each ideal \( A \) with
\( A = A_v \) (or \( A = vA \)) is left and right projective.

**Question 5.3.** (1) Study the structure of divisorial ideals of a generalized
hereditary ring.
(2) Find out a necessary and sufficient conditions for \( R[t; \sigma, \delta] \) (\( R[Xt; \sigma, \delta] \)) to
be generalized hereditary in terms of the properties: the based ring \( R, \sigma, \delta \), and
describe all divisorial ideals.

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**Note:**
A right ideal \( a \) is projective \( \iff \) \( a(R : a)_l = O_l(a) \).

**Definition.** Let \( H \) be a semigroup with the quotient group \( q(H) \).
(1) \( H \) is called **hereditary**
\( \iff \) Each ideal \( a \) is right and left projective, that is: \( a(R : a)_l = O_l(a) \) and \( (R : a)_r \cdot a = O_r(a) \).
(2) $H$ is generalized hereditary if each divisorial ideal is right and left projective.

**Question 5.4.** Is it possible to obtain the arithmetical ideal theory in hereditary and generalized hereditary semigroup?