# Ore and Ore-Rees rings which are maximal orders

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$R$ : Noetherian prime ring with quotient ring $Q$

$\sigma$ : an automorphism of $R$ and $\delta$ : a left $\sigma$ -derivation, that is
(i) $\delta(a+b) = \delta(a) + \delta(b)$ and
(ii) $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$.

**Ore extension of** $R$ is:
$R[t;\sigma,\delta] = \{f(t) = a_n t^n + \cdots + a_0 | a_i \in R\}$
, where $t$ is an indeterminate and the multiplication is defined by:

$$ta = \sigma(a)t + \delta(a),\ (a \in R).$$

• $X$ : invertible ideal of $R$, that is
$XX^{-1} = R = X^{-1}X$

• $S = R[Xt;,\sigma,\delta] = R \oplus Xt \oplus \cdots \oplus X^n t^n \oplus \dots$
,which is a subset of $R[t;\sigma,\delta]$
It is easy to see that $S$ is a ring if and only if $\sigma(X) = X$.
$S$ is called an **Ore-Rees ring** if $S$ is a ring

## 1 General Theory of Ore-Rees Rings

**Notation:** For any $R$-ideal $\mathfrak{a}$, $(R:\mathfrak{a})_l = \{q \in Q | q\mathfrak{a} \subseteq R\}$, $(R:\mathfrak{a})_r = \{q \in Q | \mathfrak{a}q \subseteq R\}$,
$\mathfrak{a}_v = (R:(R:\mathfrak{a})_l)_r \supseteq \mathfrak{a}$. and ${}_v\mathfrak{a} = (R:(R:\mathfrak{a})_r)_l \supseteq \mathfrak{a}$.
If ${}_v\mathfrak{a} = \mathfrak{a}$ ($\mathfrak{a} = \mathfrak{a}_v$), then $\mathfrak{a}$ is called a left divisorial (right divisorial) ideal.
If ${}_v\mathfrak{a} = \mathfrak{a} = \mathfrak{a}_v$, then $\mathfrak{a}$ is just called a **divisorial** $R$-ideal .

**Definition** $R$ is a **maximal oder** if $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of $R$
,where $O_l(\mathfrak{a}) = \{q \in Q | q\mathfrak{a} \subseteq \mathfrak{a}\}$, the *left order of* $\mathfrak{a}$ and $O_r(\mathfrak{a}) = \{q \in Q | \mathfrak{a}q \subseteq \mathfrak{a}\}$, the *right order* of $\mathfrak{a}$.
$\Leftrightarrow$
Any divisorial $R$-ideal $\mathfrak{a}$ is *divisorially invertible*, that is $(\mathfrak{a}\mathfrak{a}^{-1})_v = R = {}_v(\mathfrak{a}^{-1}\mathfrak{a})$
$\Leftrightarrow$
$D(R) = \{\mathfrak{a}$: divisorial $R$-ideal in $Q$ $\}$

is an Abelian group under the multiplication "$\circ$"; $\mathfrak{a} \circ \mathfrak{b} = (\mathfrak{a}\mathfrak{b})_v$.

### Examples of Maximal orders

(1) In case of commutative domains, maximal orders $\Longleftrightarrow$ completely integrally closed domains
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(2) In case of non-commutative rings,

(a) Algebra case:

$$\begin{tikzcd} & Q \arrow[d, dash] \\ K \arrow[ur] & R \\ D \arrow[u, dash] \arrow[ur] & \end{tikzcd}$$

Let $Q$ be a simple Artinian ring with $[Q : K] < \infty$, where $K = \mathbb{Z}(Q)$ is the center of $Q$. A subring $R$ of $Q$ with the center $D$ is called a $D$-**order** in $Q$ if the following are satisfied:
(i) $K = Q(D)$, the quotient field of $D$ and $KR = Q$.
(ii) Every element of $R$ is integral over $D$.
Then:
(1) There always exists a maximal $D$-order in $Q$.

(2) If $D$ is a Dedekind domain, then any maximal order is a non-commutative Dedekind ring (see, I. Reiner: Maximal orders, Academic Press, 1975).

(3) If $D$ is a Krull domain, then any maximal order is a non-commutative Krull ring (see, e.g. H.Marubayashi and F. Van Oysteayen: Prime Divisors and Noncommutative Valuation Theory, Lecture Notes in Math. 2059, Springer, 2012).

(b) If $R$ is a maximal order, then the Ore-extension $R[t; \sigma, \delta]$ and skew formal power series ring $R[[t; \sigma]]$ are maximal orders.

(c) Non-commutative Krull rings , unique factorization rings and regular rings are all maximal orders.

(e) The class of maximal orders are occupied the important parts in enveloping algebras , crossed product algebras(including group rings) and semi-group algebras (E.Jespers and J. Okininski's book: Noetherian Smigroup Algebras).

• $(\sigma, \delta)$ are naturally extended to the automorphism $\sigma$ of $Q$ and the left $\sigma$-derivation on $Q$ by $\sigma(ac^{-1}) = \sigma(a)\sigma(c)^{-1}$ and $\delta(c^{-1}) = -\sigma(c^{-1})\delta(c)c^{-1}$ for any $a, c \in R$ and $c$ is a regular element in $R$.

Let $T = Q[t; \sigma, \delta]$, Ore extension of $Q$. It is well known that
(i) $T$ is a **principal ideal ring** and
(ii) $AT$ is an ideal of $T$ for each ideal $A$ of $S$ (the proof is not difficult).

**Theorem 1.1.** If $R$ is a maximal order, then $S = R[Xt; \sigma, \delta]$ is a maximal order. But the converse is not true.

The outline of the proof:

Let $A$ be an ideal of $S$. We need to prove: $O_l(A) = S$ (it is clear that $O_l(A) \supseteq S$). To show the converse inclusion, Let $q \in O_l(A)$. We have:
(i) $q \in T$, that is $q = q_l t^l + \cdots + q_0$ $(q_i \in Q)$
(ii) $C_n(A) = \{\ a \in R | \exists h(t) = at^n + \cdots + a_0 \in A\ \} \cup (0)$. Then:

$$X^k \sigma^k(C_m(A)) = C_{m+k}(A)$$

for some $m$ and for any $k$.

• In order to study the properties of divisorial ideals of $S = R[Xt, \sigma, \delta]$, we need the following concept:

**Definition.** An $R$-ideal $\mathfrak{a}$ in $Q$ is called $(\sigma, \delta; X)$-*stable* if $X\sigma(\mathfrak{a}) = \mathfrak{a}X$ and $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$.
In case $\delta = 0$, $\mathfrak{a}$ is called $(\sigma; X)$-*invariant*.

**Lemma 1.2.** Let $\mathfrak{a}$ be an ideal of $R$. Then $A = \mathfrak{a}[Xt; \sigma, \delta]$ is an ideal of $S$ if and only if $\mathfrak{a}$ is $(\sigma, \delta; X)$-stable.

**Definition.** $R$ is a $(\sigma, \delta; X)$-**maximal order** in $Q$ if $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for any $(\sigma, \delta; X)$-stable ideal $\mathfrak{a}$.

**Proposition 1.3.** If $R$ is a $(\sigma, \delta; X)$-maximal order, then $D_{(\sigma,\delta,X)}(R)= \{$ $\mathfrak{a}$ : $(\sigma, \delta; X)$-stable and divisorial $R$-ideals $\}$ is an Abelian group generated by maximal $(\sigma, \delta; X)$-stable divisorial ideals.

**Proposition 1.4.** Suppose $R$ is a $(\sigma, \delta; X)$-maximal order. Let $P$ be a prime ideal of $S$ such that $P \cap R = \mathfrak{p}$ is $(\sigma, \delta; X)$**-stable** and $P$ is divisorial. Then $P = \mathfrak{p}[Xt; \sigma, \delta]$.

**2 Differencial Rees ring**

In case $\sigma = 1$ and $\delta \neq 0$,
$S = R[Xt; 1, \delta] = R[Xt; \delta]$ is called a **differencial Rees ring**.

• An $R$-ideal $\mathfrak{a}$ in $Q$ is called $(\delta; X)$-*stable* if $X\mathfrak{a} = \mathfrak{a}X$ and $X\delta(\mathfrak{a}) \subseteq \mathfrak{a}$.

• $R$ is a $(\delta; X)$-**maximal order** if $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for all $(\delta; X)$-stable ideal $\mathfrak{a}$ of $R$.

**Theorem 2.1.** $S = R[Xt; \delta]$ is a maximal order if and only if $R$ is a $(\delta; X)$-maximal order.

• The outline of the proof: If $R$ is a $(\delta; X)$-maximal order, then $S = R[Xt; \delta]$ is a maximal order.

**Lemma 2.2.** Let $P$ be a prime ideal of $S$ with $P = P_v$ $(P = {}_vP)$. Then $\mathfrak{p} = P \cap R$ is $(\delta; X)$-stable.

This lemma is proved by considering two cases: either $P \supseteq X$ or $P \not\supseteq X$.

**Lemma 2.3** Suppose $R$ is a $(\delta; X)$-maximal order. Let $A$ be a divisorial ideal of $S$. If $\mathfrak{a} = A \cap R \neq (0)$. Then
(1) $\mathfrak{a}$ is a $(\delta; X)$-stable divisorial ideal and
(2) $A = \mathfrak{a}[Xt; \delta]$. In particular, $A$ is divisorially invertible.

The outline of the proof.

Put $\mathfrak{B} = \{A$: ideal $\mid A \cap R \neq 0$ and $A = A_v\ \}$.
If $P$ is maximal in $\mathfrak{B}$, then $P$ is a prime ideal and so $P = \mathfrak{p}[Xt, \delta]$ by Proposition 1.4 and Lemma 2.2.

**Lemma 2.4.** Suppose $R$ is a $(\delta; X)$-maximal order. Let $A$ be a divisorial ideal with $A \cap R = (0)$. Then $A$ is a divisorially invertible ideal.

This lemma is proved by the following way: $A(S : A)_r \cap R \neq (0)$ and, by Lemma 2.3, $(A(S : A)_r)_v = \mathfrak{a}[Xt; \delta]$,where $\mathfrak{a} = (A(S : A)_r \cap R)_v$

$\Longrightarrow ((A(S:A)_r)_v \mathfrak{a}^{-1}]Xt;\delta])_v = S$. Hence $A$ is divisorially invertible.

**Theorem 2.5.** Suppose $R$ is a $(\delta;X)$-maximal order. Then any divisorial $S$- ideal $A$ is of the form:

$$A = w\mathfrak{a}[Xt;\delta]$$

,where $w \in \mathbb{Z}(Q(T))$ = the center of $Q(T)$ and $\mathfrak{a}$ is a $(\delta;X)$-stable divisorial $R$-ideal.

This is easily proved by Lemmas 2.3, 2.4 and the fact: any ideal $A'$ of $T$ is of the form: $A' = wT$ for some $w \in \mathbb{Z}(T)$ (see: G.Cauchon's Ph.D thesis, 1977).

## 3 Skew Rees rings

In case $\delta = 0$, $S = R[Xt;\sigma,0] = R[Xt;\sigma]$ is called a **skew Rees ring**.

• An ideal $\mathfrak{a}$ of $R$ is $(\sigma;X)$-**invariant** if
$X\sigma(\mathfrak{a}) = \mathfrak{a}X$
• $R$ is a $(\sigma;X)$-**maximal order** if
$O_r(\mathfrak{a}) = R = O_l(\mathfrak{a})$ for all $(\sigma;X)$-invariant ideal $\mathfrak{a}$ of $R$.

**Theorem 3.1.** $S = R[Xt;\sigma]$ is a maximal order if and only if $R$ is a $(\sigma;X)$-maximal order.

**Theorem 3.2.** Suppose $R$ is a $(\sigma;X)$-maximal order. Then any divisorial $S$- ideal $A$ is of the form:

$$A = t^n w\mathfrak{a}[Xt;\sigma]$$

,where $w \in \mathbb{Z}(Q(T)$ = the center of $Q(T)$ and $\mathfrak{a}$ is a divisorial $(\sigma;X)$-invariant $R$-ideal , $n$ is an integer and $Q(T)$ is the quotient ring of $T = Q[t;\sigma]$.

## 4 Unique factorization rings (UFRs) and krull rings

**Definition.** (1) $R$ is a **UFR** in the sense of Chatters and Jordan(C-J) if any non-zero prime ideal contains a principal prime ideal (see: Kaplansky's book).

(2) $R$ is a **UFR** in the sense of mine if for any prime ideal $P$ with $P = P_v$ ($P =_v P$) is principal, that is $aR = P = Ra$ for some $a \in R$( see: Samuel's Lecture from Tata Institute).

**Definition** $R$ is called a **Krull ring** in the sense of Chamarie if
(1) $R$ is a maximal order.
(2) $R$ satisfies the a.c.c. on one sided "closed" ideals.

$\Longleftrightarrow$
(1) $R = \cap R_P \cap S(R)$, where $R_P$ is a local Dedekind prime ring ($P$ runs over all prime divisorial ideals of $R$) and $S(R) = \cup A^{-1}$ **is a divisorially simple** ($A$ runs over all ideals of $R$)
. (2) For any regular element $c$ in $R$, $cR_P = R_P$ for almost all $P$.

(i)
$$\begin{tikzcd}
\text{(UFRs of C-J)} \arrow[r, Rightarrow] \arrow[d] & \text{(Krull rings of mine)} \arrow[d] \\
\text{(UFRs of mine)} \arrow[r, Rightarrow] & \text{(Krull rings of Chamarie)}
\end{tikzcd}$$

(ii) If $R$ is a UFR in the sense of mine, then so are $R[t,\sigma]$ and $R[t,\delta]$. But if $R$ is a UFR in the sense of (C-J), then both $R[t,\sigma]$ and $R[t,\delta]$ are not necessary to be UFRs in the sense of (C-J).

## 5 Open questions and Remarks

**Question 5.1.** (Long standing open question) Find out a necessary and sufficient conditions for Ore extension $R[t;\sigma,\delta]$ to be a maximal order ( a UFR) and describe the structure of divisorial ideals (see: M.Chamarie, Anneaux de Krull non commutative, these, 1981).

**Question 5.2.** Find out a necessary and sufficient conditions for $S = R[Xt,\sigma,\delta]$ to be a maximal order (a UFR) and describe the structure of divisorial ideals of $S$.

• In case $\delta = 0$, the skew Rees ring $R[Xt;\sigma]$ is a UFR $\Longleftrightarrow$
$X$ is principal and $R$ is a $(\sigma;X)$-UFR, that is any $(\sigma;X)$-invariant divisorial ideal is principal.

• In case $\sigma = 1$, the differential Rees ring $R[Xt;\delta]$ is a UFR
$\Longleftrightarrow$
$X$ is principal and $R$ is a $(\delta;X)$-UFR, that is any $(\delta;X)$-stable divisorial ideal is principal.

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$$\begin{tikzcd}
\text{(Dedekind rings)} \arrow[r] \arrow[dr] & \text{(Asano rings)} \\
& \text{(Hereditary rings)}
\end{tikzcd}$$

**Definition.** (1) $R$ is **Dedekind** $\Longleftrightarrow$
it is a maximal order and any one sided ideal is projective.
(2) $R$ is **Asano** if any ideal is invertible.
(3) $R$ is **hereditary** if any one sided ideal is projective.

The following is the simplest examples such that these three concepts are different:

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \text{ Dedekind}$$

$$|$$

$$S = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \text{ hereditary and but neither Dedekind nor Asano}$$

.where $\mathbb{Z}$ is the ring of integers and $p$ is a prime number.

$$M = \begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \ N = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & p\mathbb{Z} \end{pmatrix} \Rightarrow M = M^2$$

and $N = N^2$.

Let $R$ be a Noetherian simple ring but not Artinian (e.g. the nth- Weyl algebra with char $R = 0$ for any $n$) and $R[t]$, the polynomial over $R$.
Then $R[t]$ is Asano but neither Dedekind nor hereditary.

• Let $R$ be a hereditary ring and $S = R[t]$, the polynomial ring. Then $S$ is not hereditary if $R \neq Q$.

• what kind of properties does $R[t]$ have from the arithmetical point of view?
• Any ideal $A$ of $R[t]$ with $A = A_v$ (or $A =_v A$) is left and right projective.

This fact leads us to define the following:

**Definition.** $R$ is called a **generalized hereditary** if each ideal $A$ with $A = A_v$ (or $A = {}_vA$) is left and right projective.

**Question 5.3.** (1) Study the structure of divisorial ideals of a generalized hereditary ring.
(2) Find out a necessary and sufficient conditions for $R[t; \sigma, \delta]$ ($R[Xt; \sigma, \delta)$) to be generalized hereditary in terms of the properties: the based ring $R$, $\sigma, \delta$, and describe all divisorial ideals.

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Note:
A right ideal $\mathfrak{a}$ is projective $\Leftrightarrow \mathfrak{a}(R : \mathfrak{a})_l = O_l(\mathfrak{a})$.

**Definition .** Let $H$ be a semigroup with the quotient group $q(H)$.
(1) $H$ is called **hereditary**
$\Longleftrightarrow$
Each ideal $\mathfrak{a}$ is right and left projective, that is: $\mathfrak{a}(R : \mathfrak{a})_l = O_l(\mathfrak{a})$ and $(R : \mathfrak{a})_r\mathfrak{a} = O_r(\mathfrak{a})$.

(2) $H$ is **generalized hereditary** if each divisorial ideal is right and left projective.

**Question 5.4.** Is it possible to obtain the arithmetical ideal theory in hereditary and generalized hereditary semigroup?