ATOMIC DECAY
in DOMAINS and MONOIDS

Ulrich Krause
Department of Mathematics,
University of Bremen, Germany

Arithmetic and Ideal Theory
of Rings and Semigroups
with a one–day special session dedicated to
Franz Halter–Koch on the occasion of his 70th birthday

September 22–26, 2014, Graz, Austria

1 Definition

$(M, \cdot)$ monoid, cancellative, commutative with 1

atomic decay of atom $x \in M$ into atoms x_i

$$x^m = x_1 \ldots x_n$$

$\boldsymbol{x}$ strong atom $y \mid x^m$, y atom $\Rightarrow$ y and x assoc (no decay)

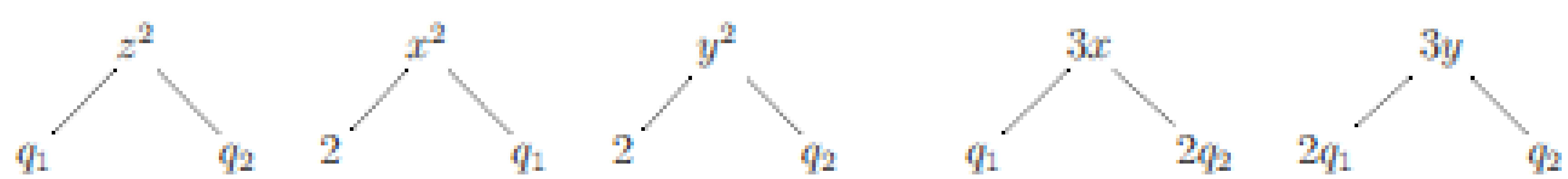
2 Examples

domain $M = \mathbb{Z}[\sqrt{-5}], \cdot$ **monoid** $M = \{x \in \mathbb{Z}_+^3 \mid 2x_1 + 5x_2 = 3x_3\}, +$

non–unique factorization

$\underset{q}{6} = \underset{z}{2} \cdot 3 = \underset{x}{(1+\sqrt{-5})} \; \underset{y}{(1-\sqrt{-5})}$ **atoms** $(3,3,7) = \underset{q_1}{(3,0,2)} + \underset{q_2}{(0,3,5)} = \underset{x}{(1,2,4)} + \underset{y}{(2,1,3)}$

decay



$2, q_1 = -2 + \sqrt{-5}, q_2 = \bar{q}_1$ **strong atoms** q_1, q_2

unique facotrization into strong atoms by decay

$$6^2 = \begin{array}{l} q^2 z^2 = 2 \cdot 2 \cdot q_1 \cdot q_2 \\ x^2 y^2 = 2 \cdot q_1 \cdot 2 \cdot q_2 \end{array} \quad 3(3,3,7) = \begin{array}{l} 3q_1 + 3q_2 \\ 3x + 3y = q_1 + 2q_2 + 2q_1 + q_2 \end{array}$$

3 Decay Theorem

$(M, \cdot)$ Krull monoid, class group $Cl(M)$ torsion group

Theorem

(i) For each non–unit $x \in M$ there exist $1 \leq m(x)$ minimal and $q(x) \in \mathbb{Z}_+$ such that

$$x^{m(x)} = \prod_{q \text{ strong atoms}} q^{q(x)}$$

and this factorization is unique up to units
(and ordering of factors).

(ii) Exp $Cl(M) = lcm\{m(x) \mid x \text{ atom}\}$

(iii) For the **decay rate** $\delta(x) = \frac{1}{m(x)} \sum_q q(x)$

$\sup\{\delta(x) \mid x \text{ atom}\} \leq k(Cl(M))$ (cross number)

Corollary

(i) $\delta(xy) = \delta(x) + \delta(y)$ for non–units

(ii) M half–factorial $\Leftrightarrow$ $\delta(x) = 1$ for all atoms

(iii) M factorial $\Leftrightarrow$ no atomic decay (all atoms strong)

(iv) Each class of $Cl(M)$ contains at least one prime divisor
$\Rightarrow \sup\{\delta(x) \mid x \text{ atom }\} = k(Cl(M))$

Examples

domain $Cl(M) = C_2, \text{Exp }(Cl(M)) = 2, k(Cl(M)) = 1$

x atom: $x^2 = qq', \delta(x) = 1, M$ half–factorial (not factorial)

monoid $Cl(M) = C_3, \text{Exp }(Cl(M)) = 3, k(Cl(M)) = 1$

x atom: $3x = q + 2q', \delta(x) = 1, M$ half–fact (not fact, $Cl(M) \neq C_2$)

Extraction $x, y \in M$ non–units

$$\lambda(x,y) = \sup\{\tfrac{m}{n} \mid x^m \text{ divides } y^n, m, n \geq 1\}$$

- For each strong atom $q \in M$ exists $l(q) \in \mathbb{Z}_+$ such that

  $$x \mapsto l(q)\lambda(q,x)$$

  is a homomorphism of the non–units of M onto $(\mathbb{Z}_+, +)$.
  By this strong atoms correspond to essential states on M.
- $\frac{q(x)}{m(x)} = \lambda(q,x), \delta(x) = \sum_q \lambda(q,x)$
- $m(x) = lcm\{l(q) \cdot [gcd\{l(q), l(q)\lambda(q,x)\}]^{-1}\}$

Question Construction of a divisor theory purely by extraction?

4 Taking roots

M Krull monoid, $Cl(M)$ torsion group

$$x^m = \prod_{q \text{ strong}} q^{q(x)} \stackrel{?}{\longrightarrow} x = \prod_{q \text{ strong}} (q^{\frac{1}{m}})^{q(x)}$$

$\mathbb{Z}_+ \times M$ with $(m, x) \sim (n, y)$ iff $x^n = y^m$, $\sim$ equivalence rel
$R(M) = \mathbb{Z}_+ \times M/_\sim$, class $[m, x]$ of (m, x) $\boldsymbol{m}$**–th root of** $\boldsymbol{x}$
$[m, x] \cdot [n, y] = [mn, x^n y^m]$ well–defined
$(R(M), \cdot)$ **monoid of roots of** $\boldsymbol{M}$, comm, unity $[1, 1]$,
(almost cancell)

Then for non–unit $x \in M$
$\boldsymbol{[1, x] = \prod [m, q]^{q(x)}}$, $[m, q]$ m–th root of strong atom q

$[m, q]$ unique up to $[m, u]$ m–th root of unit $u \in M$

5 Domains

D Krull domain, $Cl(D)$ torsion, $M = D^\bullet$
In particular: D maximal order O of an algebraic number field

J. Kaczorowski for $D = O; \alpha \in O$ irreducible is

completely irreducible iff α^n has a unique factorization for each n

F. Halter–Koch for D arithmetical Dedekind: $\pi \in D$ irred. is

absolutely irreducible iff $\pi \mid \alpha\beta, \pi \neq \alpha \Rightarrow \pi \mid \beta^n$ some n

Characterization of strong atoms

(i) D integral domain, $q \in D$ atom

q strong $\Leftrightarrow$ q can be separated from any non–ass. atom x

by a prime ideal P, i.e. $q \notin P, x \in P$

(ii) D Krull domain, $Cl(D)$ torsion

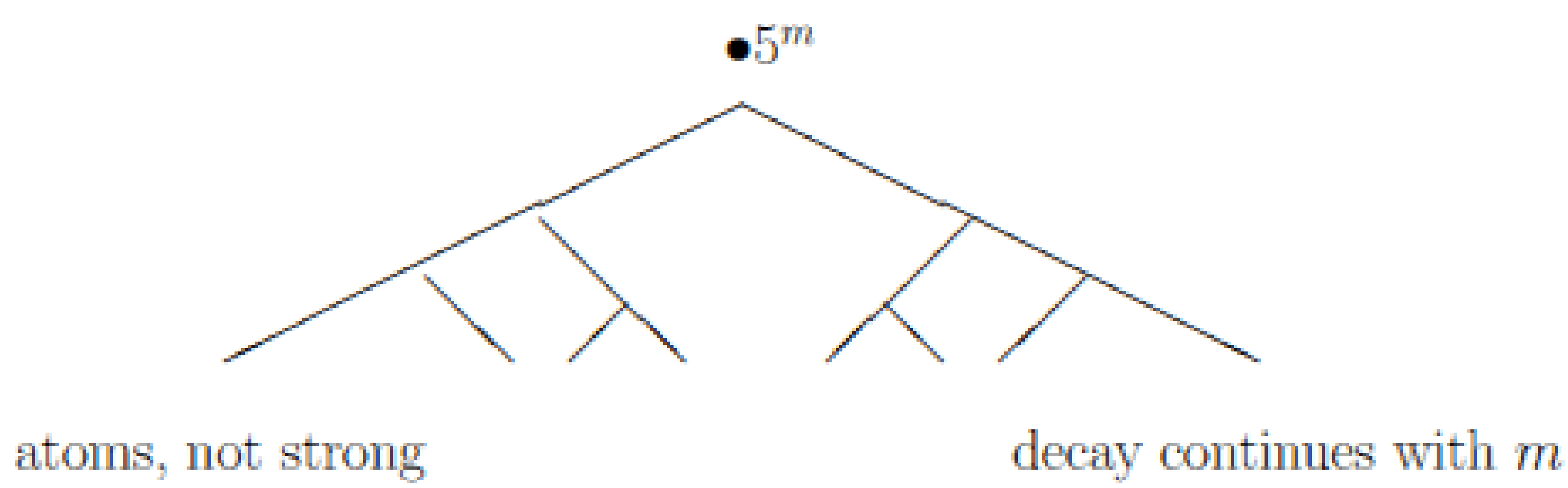
For an atom $q \in D$ are equivalent

- q strong
- q absolutely/completely irreducible (primary)
- $v(x) = \lambda(q, x)$ defines an essential valuation

M monoid prime $\Rightarrow$ strong atom $\Rightarrow$ atom, none reversible

Theorem on atomic decay fails if there are too few or too "soft" strong atoms

Example $D = \mathbb{Z}[5\sqrt{-1}], \cdot$ too few strong atoms



Atomic decay in maximal order O

Decay Theorem: $x^{m(x)} = \prod q^{q(x)}$, q strong atoms (equiv ...)

Taking roots $\mathcal{R} = \{\alpha \in \mathbb{C} \mid \alpha^n \in O \text{ some } n \geq 1\}$, monoid
$\psi\colon \mathcal{R} \to R(O^\bullet), \psi(\alpha) = [n, x]$ for $\alpha^n = x$ well–def., surj. hom.
Decay Theorem: $[1, x] = \prod [m, q]^{q(x)}, m = m(x)$
$\Rightarrow \boldsymbol{x = \prod \alpha_q^{q(x)}}$, $\alpha_q^m = q, \alpha_q$ m–th root of strong atom q
α_q unique up to m–th root of units $u \in O$ (and ordering)

- **T. Skolem, F. Halter–Koch** (structure theorem for semigroups)

  reason: group structure of an algebraic number field obtained since a *subgroup* of the free divisor *group* is free

- **E. Hecke** takes roots (for a particular example) which he interpretes as Kummers ideal numbers.

Question Do the roots α_q of strong atoms correspond to Jacobi's "wahre complexe Primzahlen"?

6 Diophantine monoids

M Krull monoid, $M \subseteq \mathbb{Z}_+^n$ add, $Cl(M)$ torsion
$x \in M$ strong atom: $y \leq mx, y$ atom, $m \geq 1 \Rightarrow y = x$
Factorization by atomic decay, $0 \neq x \in M$

$$m(x)x = \sum_{q \text{ strong}} q(x)\, q \text{ unique (up to ordering)}$$

Roots $(m, x) \sim (n, y) \Leftrightarrow nx = my$, write $\frac{x}{m} = \frac{y}{n}$
yields $x = \sum_{q \text{ strong}} q(x)(\frac{q}{m})$
In partiuclar, M **Diophantine monoid**,
$M = \{x \in \mathbb{Z}_+^n \mid Ax = 0\}, A \in \mathbb{Z}^{r\times n}$ system of r linear Dioph. equations in n **nonnegative** unknowns.
(equiv. Krullmonoid with finitely many essential states/finitely generated)

R. Stanley

- $x \in M$ **fundamental** $x = y + z, y, z \in M \Rightarrow y = 0$ or $z = 0$
- x **completely fundamental** $mx = y + z, m \geq 1, y, z \in M \Rightarrow y = sx, 0 \leq s \leq m$

Obviously, fundamental $\leftrightarrow$ atom, completely f. $\leftrightarrow$ strong atom
To determine the solution set M is extremely different
Example Magic squares: Stanley develops theory, using generating functions, to determine the number of squares. Known only for small n, many conjectures.

1 equation in n unknowns

$M = \{x \in \mathbb{Z}_+^n \mid a_1x_1 + \cdots + a_nx_n = 0\}, a_i \in \mathbb{Z}, gcd\{a_i\} = 1$
Krull monoid, divisor theory, $Cl(M)$ maybe **not** torsion

Example $M = \{x \in \mathbb{Z}_+^4 \mid x_1 + x_2 - x_3 - x_4 = 0\}, Cl(M) = \mathbb{Z}$
strong atoms $q_1 = (1,0,1,0), q_2 = (0,1,0,1), q_3 = (1,0,0,1)$,
$q_4 = (0,1,1,0)$
Decay theorem **not** applicable, indeed $q_1 + q_2 = q_3 + q_4$.
All atoms are strong but "soft" (not primary).

Consider
$M = \{x \in \mathbb{Z}_+^n \mid a_1x_1 + \cdots + a_{n-1}\, x_{n-1} = a_nx_n\}, a_i \in \mathbb{Z}_+$

$Cl(M) = \mathbb{Z}_b, b = \frac{a_n}{\prod_{i=1}^{n-1} b_i}, b_i = gcd\{a_j \mid j \neq i\}$

Decay Theorem $m(x)x = \sum_{q \text{ strong}} q(x)q$, uniqueness

The strong atoms are, for $1 \leq i \leq n-1$,
$q_i = \frac{1}{gcd\{a_i,a_n\}}(a_n e_i + a_i e_n), e_i$ i–th unit vector in $\mathbb{Z}^n$.

Finding the solutions still difficult!
Wanted: Unique description by parameters

- M factorial $\Leftrightarrow a_n = \prod_{i=1}^{n-1} gcd\{a_j \mid j \neq i\}$

  Solutions $x = \sum_{q \text{ strong}} n(q)q, n(q) \in \mathbb{Z}_+$
- M half–factorial $\Leftrightarrow m(x) = \sum_{q \text{ strong}} q(x)$, atoms x

Questions Half–factoriality in terms of the a_i?
Which decays of atoms are possible?

1 equation in 3 unkowns

$M = \{x \in \mathbb{Z}_+^3 \mid a_1x_1 + a_2x_2 = a_3x_3\}, gcd\{a_i \mid 1 \le i \le 3\} = 1$

E.B. Elliott 1903 for $a_3 = 1, 2, \ldots 10$ using generating functions.
M **factorial** iff $a_3 = d_1d_2, d_i = gcd\{a_i, a_3\}$
solutions $x = \left(m\frac{a_3}{d_1}, n\frac{a_3}{d_2}, m\frac{a_1}{d_1} + n\frac{a_2}{d_2}\right), m, n \in \mathbb{Z}_+$

M **half–factorial** iff $a_3 \mid a_id_j - a_jd_i, 1 \le i, j \le 2$
normalizing the equation: $gcd\{a_i, a_j\} = 1, i \ne j, a_3 \mid a_1 - a_2$

$q_1 = (a_3, 0, a_1), q_2 = (0, a_3, a_2)$, by atomic decay, $Cl(M) = \mathbb{Z}_{a_3}$

$rx = k_1q_1 + k_2q_2, k_1 + k_2 = r, r \mid a_3$ for atom x

$\Rightarrow$ atoms given by $\left(k, a_3 - k, a_2 + k\ \frac{a_1 - a_2}{a_3}\right)$

with parameters, $0 \le k \le a_3$.

Example $a_1x_1 + a_2x_2 = 3x_3$

factorial $x = (3m, 3n, ma_1 + na_2), m, n \in \mathbb{Z}_+$

half–factorial atoms are $x = (k, 3-k, a_2 + k\frac{a_1-a_2}{3})$
initial example $2x_1 + 5x_2 = 3x_2$
atoms $x = (k, 3-k, 5-k) 0 \leq k \leq 3$,
just as stated

not half–factorial $rx = k_1q_1 + k_2q_2 = (3k_1, 3k_2, k_1a_1 + k_2a_2)$

$r \mid 3, k_1 + k_2 \leq r(\delta(x) \leq k(Cl(M)) = k(\mathbb{Z}_3) = 1)$

$k_1a_1 + k_2a_2 = (k_1 + k_2)a_2 + (a_1 - a_2)k_1 \Rightarrow k_1 + k_2 < 3$
since $3 \nmid a_1 - a_2$

$\Rightarrow k_1 + k_2 \leq 2$. $k_1 = 0$ or $k_2 = 0$ yields q_2 or q_1

$\Rightarrow$ remaining case $k_1 = k_2 = 1 \Rightarrow 3x = q_1 + q_2 = (3, 3, a_1 + a_2)$

$\Rightarrow x = \left(1, 1, \frac{a_1+a_2}{3}\right)$

For example, $x_1 + 2x_2 = 3x_3$ not half–factorial
since $3(1,1,1) = (3,0,1) + (0,3,2)$ – decay rate of $(1,1,1)$ is $\frac{2}{3} < 1$.