Piecewise w-Noetherian domains and their applications

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Arithmetic and Ideal Theory of Rings and Semigroups
Graz, Austria
September 26, 2014

Outline

Star operations and related domains

- R : Integral domain with quotient field K.
- $\mathcal{F}(R)$: The set of nonzero fractional ideals of R.

Star operation

A $*$*-operation* (star operation) on R is a mapping $A \mapsto A_*$ from $\mathcal{F}(R)$ to $\mathcal{F}(R)$ which satisfies the following conditions for all $a \in K \setminus \{0\}$ and $A, B \in \mathcal{F}(R)$:

- $(a)_* = (a)$ and $(aA)_* = aA_*$,
- $A \subseteq A_*$; if $A \subseteq B$, then $A_* \subseteq B_*$, and
- $(A_*)_* = A_*$.

An $A \in \mathcal{F}(R)$ is called a $*$*-ideal* if $A_* = A$ and A is called $*$*-finite* if $A = B_*$ for some f. g. $B \in \mathcal{F}(R)$. A is said to be $*$*-invertible* if $(AB)_* = R$ for some $B \in \mathcal{F}(R)$.

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Star operations and related domains

Examples of star operations

For $A \in \mathcal{F}(R)$,

- d-operation : $A_d := A$;
- v-operation : $A_v := A \mapsto (A^{-1})^{-1}$, where $A^{-1} = R :_K A$;
- t-operation : $A_t := \cup\{J_v | J \subseteq A$ with $J \in \mathcal{F}(R)$ f.g.$\}$;
- w-operation : $A_w := \{x \in K \mid Jx \subseteq A$ for some $J \in \mathrm{GV}(R)\}$, where $J \in \mathrm{GV}(R)$ if J is a f.g. ideal of R with $J^{-1} = R$.
- $A_d \subseteq A_w \subseteq A_t \subseteq A_v$.
- $R \subseteq T$ *t-linked extension* if $J \in GV(R)$ implies that $JT \in GV(T)$.
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Related domains

- Recall that an integral domain R is called a *Prüfer v-multiplication domain* (for short, *PvMD*) if A_v (equivalently A^{-1}) is t-invertible for every f.g. ideal A of R.
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Definition

A commutative ring R with identity is said to be *piecewise Noetherian* if (i) the set of prime ideals of R satisfies the ACC; (ii) R has the ACC on P-primary ideals for each prime ideal P; and (iii) each ideal has only finitely many prime ideals minimal over it.

Theorem 1.4.

If R is a piecewise Noetherian ring, then a flat overring of R is also piecewise Noetherian.

Corollary 1.5.

Let R be an integral domain and let S be a multiplicative set of R. If R is piecewise Noetherian, then R_S is also piecewise Noetherian. In particular, R_P is piecewise Noetherian for all prime ideals P of R.

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Let R be an integral domain, $R[X]$ be the polynomial ring over R, and $S = \{f \in R[X] \mid c(f) = R\}$. Then $R(X) = R[X]_S$, called the *Nagata ring* of R, is an overring of $R[X]$. The next result is a piecewise Noetherian domain analogue of the well-known fact that R is Noetherian if and only if $R[X]$ is Noetherian, if and only if $R(X)$ is Noetherian.

Corollary 1.6

The following are equivalent for an integral domain R.

1. R is piecewise Noetherian.
2. $R[X]$ is piecewise Noetherian.
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Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of overrings of an integral domain R such that $R = \bigcap_{\alpha\in\Lambda} R_\alpha$. We say that the intersection $R = \bigcap_{\alpha\in\Lambda} R_\alpha$ is of *finite character* if each nonzero element of R is a unit in R_α for all but a finite number of R_α.

Theorem 1.7.

Let R be an integral domain and let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of flat overrings of R such that $R = \bigcap_{\alpha\in\Lambda} R_\alpha$ is of finite character. Assume that we have $I = \bigcap_{\alpha\in\Lambda} IR_\alpha$ for all ideals I of R. Then each R_α is piecewise Noetherian if and only if R is piecewise Noetherian.

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An *almost Dedekind domain* R is an integral domain such that R_M is a principal ideal domain for all maximal ideals M of R.

Corollary 1.8.

Let R be an integral domain of finite character. Then R is piecewise Noetherian if and only if R_M is piecewise Noetherian for all maximal ideals M of R.

Recall that a valuation domain is *strongly discrete* if it has no non-zero idempotent prime ideal; a *strongly discrete Prüfer domain* is a domain whose localization at any nonzero prime ideal is a strongly discrete valuation domain; an integral domain R is a *generalized Dedekind domain* if it is a strongly discrete Prüfer domain and every prime ideal of R is the radical of a finitely generated ideal.

Corollary 1.9.

If R is a strongly discrete Prüfer domain of finite character, then R is piecewise Noetherian, and hence generalized Dedekind.

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Let T be a quasi-local integral domain with maximal ideal M, $Q = T/M$, $\phi : T \to Q$ the canonical ring homomorphism, D a proper subring of Q, and $R = \phi^{-1}(D)$ the pullback.

$$\begin{array}{ccc} R = \phi^{-1}(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ \phi\ } & Q = T/M \end{array}$$

We shall refer to R as a pullback of type ($\square$). Then R is a subring of T, isomorphic to a fiber product $T \times_Q D$. It is well known that M is a prime ideal of R, therefore comparable to the prime ideals of R; any prime ideal of R contained in M is a prime ideal of T; M is a t-ideal of R; and $D = R/M$.

Theorem 1.10.

Consider a pullback of type ($\square$). Then R is piecewise Noetherian if and only if T and D are piecewise Noetherian.

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Corollary 1.11.

Let K be the quotient field of an integral domain D and $R = D + XK[\![X]\!]$. Then R is a piecewise Noetherian ring if and only if D is a piecewise Noetherian ring.

Let R and T be two rings, let J be an ideal of T and let $f : R \to T$ be a ring homomorphism. In this setting, we can consider the following subring of $R \times T$:

$$R \bowtie^f J := \{(a, f(a) + j) \mid a \in R, j \in J\},$$

which is called the *amalgamation of R with T along J with respect to f* (introduced and studied by D'Anna, Finocchiaro, and Fontana).

It was shown that the ring $R \bowtie^f J$ has Noetherian spectrum if and only if R and $f(R) + J$ have Noetherian spectrum. In particular, if T has Noetherian spectrum, then $R \bowtie^f J$ has Noetherian spectrum if and only R has Noetherian spectrum. Among other things, it is shown that the following canonical isomorphisms hold:

$$\frac{R \bowtie^f J}{\{0\} \times J} \cong R \qquad \text{and} \qquad \frac{R \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(R) + J.$$

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Theorem 1.13.

The ring $R \bowtie^f J$ is piecewise Noetherian if and only if R and $f(R) + J$ are piecewise Noetherian. In particular, if T is piecewise Noetherian, then $R \bowtie^f J$ is piecewise Noetherian if and only R is piecewise Noetherian.

Let R be an integral domain. We say that R is a *piecewise w-Noetherian domain* if (i) R satisfies the ACC on prime w-ideals; (ii) R has the ACC on P-primary ideals for each prime w-ideal P; and (iii) each w-ideal has only finitely many prime ideals minimal over it. By definition, piecewise Noetherian domains and SM domains are piecewise w-Noetherian domains. The notion of a piecewise w-Noetherian domain was introduced in [El-BKW], where the authors called such an integral domain a piecewise strong Mori domain.

Lemma 2.1.

Let R be a piecewise w-Noetherian domain and let P be a prime w-ideal of R. Then R_P is a piecewise Noetherian domain.

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Lemma 2.1.

Let R be a piecewise w-Noetherian domain and let P be a prime w-ideal of R. Then R_P is a piecewise Noetherian domain.

We say that an overring T of an integral domain R is *t-flat* over R if $T_M = R_{M \cap R}$ for all maximal w-ideals M of R. Clearly, a flat overring is t-flat. Also, if Q is a prime w-ideal of a t-flat overring T of R, then $Q \cap R = (Q \cap R)_w \subsetneq R$.

Theorem 2.3

If T is a t-flat overring of a piecewise w-Noetherian domain R, then T is a piecewise w-Noetherian domain.

Theorem 2.5

Let R be a piecewise w-Noetherian domain of w-finite character and let M be a maximal w-ideal of R. Then M is of w-finite type.

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Theorem 2.7

If R is of w-finite character, then R is a piecewise w-Noetherian domain if and only if R_M is a piecewise Noetherian domain for each maximal w-ideal M of R.

Recall that a *strongly discrete PvMD* is a domain whose localization at any nonzero prime t-ideal is a strongly discrete valuation domain. El Baghdadi introduced the concept of genealized Krull domains as the t-operation version of generalized Dedekind domains as follows: An integral domain R is a *generalized Krull domain* if it is a strongly discrete PvMD and every prime t-ideal of R is the radical of a finite type t-ideal.

Corollary 2.8

If R is a strongly discrete PvMD of w-finite character, then R is a piecewise w-Noetherian domain, and hence a generalized Krull domain.

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If R is of w-finite character, then R is a piecewise w-Noetherian domain if and only if R_M is a piecewise Noetherian domain for each maximal w-ideal M of R.

Recall that a *strongly discrete PvMD* is a domain whose localization at any nonzero prime t-ideal is a strongly discrete valuation domain. El Baghdadi introduced the concept of genealized Krull domains as the t-operation version of generalized Dedekind domains as follows: An integral domain R is a *generalized Krull domain* if it is a strongly discrete PvMD and every prime t-ideal of R is the radical of a finite type t-ideal.

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If R is a strongly discrete PvMD of w-finite character, then R is a piecewise w-Noetherian domain, and hence a generalized Krull domain.

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Let w-$\mathrm{Spec}(R)$ be the set of prime w-ideals of an integral domain R. Kim *et al* defined R to have *strong Mori spectrum* if it satisfies the descending chain condition on the sets of the form $W(I) := \{P \in w\text{-}\mathrm{Spec}(R) \mid I \subseteq P\}$, where I runs over w-ideals of R (or equivalently, the induced topology on w-$\mathrm{Spec}(R)$ by the Zariski topology on $\mathrm{Spec}(R)$ is Noetherian).

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The *t-Nagata ring* $R[X]_{N_v}$ is very useful when we study ring-theoretic properties via the w-operation because $IR[X]_{N_v} \cap K = I_w$ and $I_w R[X]_{N_v} = IR[X]_{N_v}$ for all $I \in F(R)$, where $N_v = \{f \in R[X] \mid c(f)_v = R\}$. For example, R is a PvMD (resp., an SM domain) if and only if $R[X]_{N_v}$ is a Prüfer domain (resp., Noetherian domain).

Corollary 2.13.

The following are equivalent for an integral domain R.

1. R is a piecewise w-Noetherian domain.
2. $R[X]$ is a piecewise w-Noetherian domain.
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Let K be the quotient field of an integral domain D and X be an indeterminate over D. The $D + XK[X]$ construction has been very useful when we construct an easy example with prescribed properties. For example, $D + XK[X]$ is a GCD domain (resp., Bezout doman, Prüfer domain) if and only if D is. We next study the piecewise Noetherian and piecewise w-Noetherian domain properties of $D + XK[X]$.

Theorem 2.15.

Let $R = D + XK[X]$. Then R is a piecewise Noetherian domain (resp., piecewise w-Noetherian domain) if and only if D is a piecewise Noetherian domain (resp., piecewise w-Noetherian domain).

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Characterizations of SM domains

A commutative ring R is said to *satisfy (accr$_w$)* if the ascending chain of (w-)residuals of the form $N : B_1 \subseteq N : B_2 \subseteq N : B_3 \subseteq \cdots$ terminates for every w-ideal N of R and every finitely generated ideal B of R.

Theorem 2.17.

If R is of w-finite character, then R is an SM domain if (and only if) R is a piecewise w-Noetherian domain satisfying (accr$_w$).

A commutative ring R is *w-Laskerian* if each proper w-ideal of R may be expressed as a finite intersection of primary w-ideals of R.

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A piecewise w-Noetherian domain R is an SM domain if and only if R is a w-Laskerian domain.

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Consider a pullback of type ($\square$) in which T is t-local. Then R is piecewise w-Noetherian if and only if D and T are piecewise w-Noetherian.

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Let K be the quotient field of an integral domain D and $R = D + XK[\![X]\!]$. Then R is a piecewise w-Noetherian domain if and only if D is a piecewise w-Noetherian domain.

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Power series ring extensions

Example 2.22.

By utilizing an example due to M. H. Park, we give a piecewise w-Noetherian domain R such that $R[\![X]\!]$ is not piecewise w-Noetherian.

Unlike the “piecewise w-Noetherian domain" case, we do not know the answer to the following natural question.

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Is the power series ring $R[\![X]\!]$ a piecewise Noetherian ring if R is a piecewise Noetherian ring?

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Application: Test set for injectivity

We say that an R-module M has an *associated prime ideal* P if M contains a submodule isomorphic to R/P, equivalently $P = \mathrm{ann}_R(x)$ for some $x \in M$.

Lemma

If R is a piecewise w-Noetherian domain, then every nonzero w-module over R has an associated prime ideal.

It is well-known that over a commutative Noetherian ring R the set of all prime ideals of R is a *test set for injectivity*. That is, an R-module M is injective if and only if for any prime ideal $P \subseteq R$, any R-homomorphism $f : P \to M$ can be extended to R.

Theorem A

If R is a piecewise w-Noetherian domain, then the set of all prime w-ideals of R is a test set for injectivity of w-modules.

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Thanks!!

Thanks for your attention!