A new connection between additive number theory and invariant theory

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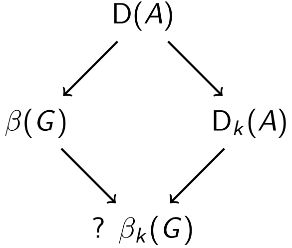
September 22, 2014

The generalized Noether number



Definition

$\beta_k(R)$ for a graded ring R is the greatest d such that $R_d \not\subseteq R_+^{k+1}$.

We write $\beta_k(G, V) := \beta_k(F[V]^G)$ and $\beta_k(G) := \sup_V \beta_k(G, V)$.
(This is finite as $\beta_k(G, V) \leq k\beta(G, V)$ while $\beta(G, V) \leq |G|$ by Noether's classical result.)

Reduction lemma for normal subgroups

Theorem (Delorme-Ordaz-Quiroz)

For any abelian groups $B \leq A$:

$$\mathsf{D}_k(A) \leq \mathsf{D}_{\mathsf{D}_k(B)}(A/B)$$

Theorem (Cz-D)

For any normal subgroup $N \triangleleft G$:

$$\beta_k(G, V) \leq \beta_{\beta_k(G/N)}(N, V)$$

Proof.

Let $F[V] = F[x_1, ..., x_n]$ and $F[V]^N = F[f_1, ..., f_r]$.
Obviously $F[V]^N$ is a G/N-module and $F[V]^G = (F[V]^N)^{G/N}$.
This means that any $g \in F[V]^G$ can be written as

$$g(x_1, ..., x_n) = p(f_1, ..., f_r)$$

for some G/N-invariant polynomial p. Let g be homogeneous of degree $\deg(g) > \beta_s(N)$ for some s. This enforces $\deg(p) > s$. Now set $s = \beta_k(G/N)$. Then p is a sum of $k+1$-fold products of non-constant G/N-invariants, whence $g \in (F[V]_+^G)^{k+1}$. □

Reduction lemma for any subgroups $H \leq G$

Theorem (Cz-D)

$$\beta_k(G, V) \leq \beta_{k[G:H]}(H, V)$$

provided that one of the following conditions holds:

- $\mathrm{char}(F) = 0$ *or* $\mathrm{char}(F) > [G : H]$
- $H \triangleleft G$ *and* $\mathrm{char}(F)$ *does not divide* $[G : H]$
- $\mathrm{char}(F)$ *does not divide* $|G|$

Open problem: the "baby Noether gap"

It is believed that in fact the above inequality holds whenever $\mathrm{char}(F)$ does not divide $[G : H]$

Lower bounds

For abelian groups $B \leq A$ it is trivial that $\mathsf{D}_k(A) \geq \mathsf{D}_k(B)$.
B. Schmid has already proved for any subgroup $H \leq G$ that:

$$\beta(G, \mathrm{Ind}_H^G V) \geq \beta(H, V)$$

A strenghtened version of her proof yields the following:

Theorem

Let $N \triangleleft G$ such that G/N is abelian. Let V be an N-module and U a G-module on which N acts trivialy. Then for any $r, s \geq 1$

$$\beta_{r+s-1}(G, \mathrm{Ind}_N^G V \oplus U) \geq \beta_r(N, V) + \mathsf{D}_s(G/N, U) - 1$$

Open problem

Can we lift the restriction that G/N is abelian? How far?

Lower bound for direct products

Theorem (Halter-Koch)

For any abelian groups A, B we have:

$$\mathsf{D}_{r+s-1}(A \times B) \geq \mathsf{D}_r(A) + \mathsf{D}_s(B) - 1$$

Theorem (Cz-D)

Let V be a G-module and U an H-module. Then for any $r, s \geq 1$

$$\beta_{r+s-1}(G \times H, V \oplus U) \geq \beta_r(G, V) + \beta_s(H, U) - 1$$

The main idea for the case $r = s = 1$ is the following:

- denote by $\mathsf{d}(A)$ the maximal length of a zero-sum free sequence over A; it is easily seen that $\mathsf{d}(A) = \mathsf{D}(A) - 1$
- let S and T be a zero-sum free sequence over A and B of length $\mathsf{d}(A)$ and $\mathsf{d}(B)$, respectively
- ST is obviously a zero-sum free sequences over $A \times B$, whence $\mathsf{d}(A \times B) \geq \mathsf{d}(A) + \mathsf{d}(B)$

How to generalize this argument for non-abelian groups?

The top degree of coinvariants

The analogue of a zero-sum free sequence for a non-abelian group is the notion of a *coinvariant*, i.e. an element of the factor ring $F[V]_G := F[V]/F[V]^G_+ F[V]$.

Observation

For any abelian group A we have:

$$D_k(G) = d_k(G) + 1$$

Theorem (Cz-K)

If V is a G-module such that $\beta_k(G, V) = \beta_k(G)$ then

$$\beta_k(F[V]^G) = \beta_k(F[V], F[V]^G) + 1$$

where $\beta_k(F[V], F[V]^G)$ gives (for $k = 1$) the top degree of the ring of coinvariants.

The growth rate of $\beta_k(G, V)$ as a function of k

We started from an easy observation that for any ring R

$$0 \leq \frac{\beta_s(R)}{s} \leq \frac{\beta_t(R)}{t} \qquad \text{for any } s \geq t \geq 1$$

Hence $\lim_{k\to\infty} \beta_k(R)/k$ exists! What is its value?

Theorem (Freeze-W. Schmid)

For any abelian group A there are integers $k_0(A)$, $\mathsf{D}_0(A)$ such that

$$\mathsf{D}_k(A) = k \exp(A) + \mathsf{D}_0(A) \qquad \textit{for any } k > k_0(A)$$

Theorem (quasi-linearity of $\beta_k(R)$)

There are some non-negative integers $k_0(R)$, $\beta_0(R)$ such that

$$\beta_k(R) = k\sigma(R) + \beta_0(R) \qquad \textit{for any } k > k_0(R)$$

Some cases where $\sigma(G)$ is known

Definition

Let $\sigma(R)$ be the smallest $d \in \mathbb{N}$ such that there are some elements $f_1, \ldots, f_r \in R$ of degree at most d whose common zero locus is $\{0\}$ — or equivalently such that R is a finite module over $F[f_1, \ldots, f_r]$.

Previously $\sigma(G)$ was studied only for linearily reductive groups.

Theorem

For an abelian group A we have $\sigma(A) = \exp(A)$.

Theorem

For $G = A \rtimes_{-1} Z_2$ we have $\sigma(G) = \exp(A)$.

Theorem

For any primes p, q such that $q \mid p - 1$ we have $\sigma(Z_p \rtimes Z_q) = p$.

This later holds also if the characteristic of the base field F equals q, as Kohls and Elmers showed.

Properties of $\sigma(G, V)$ in the non-modular case

Theorem (1)

$$\sigma(G, V_1 \oplus ... \oplus V_2) = \max_{i=1}^{n} \sigma(G, V_i)$$

Theorem (2)

$$\sigma(G, V) \leq \sigma(G/N)\sigma(N, V) \qquad \text{if } N \triangleleft G$$

Theorem (3)

$$\sigma(H, V) \leq \sigma(G, V) \leq [G : H]\sigma(H, V) \qquad \text{if } H \leq G$$

Kohls and Elmers extended the scope of this results.

A general upper bound on $\sigma(G)$

Theorem (Cz-D)

Let G be a non-cyclic group and q the smallest prime divisor of its order. Then

$$\sigma(G) \leq \frac{1}{q}|G| \qquad (1)$$

Open problem

Classify the groups with $\beta(G) \geq \frac{1}{q}|G|$! (For $q = 2$ it's done.)

Theorem (Kohls-Elmers)

Suppose the base field has caracteristic p and P is the Sylow p-subgroup of G. If G is p-nilpotent and P is not normal in G then (1) *remains true.*

Generalizing results on "short" zero-sum sequences

Definition

For any ring R let $\eta(R)$ denote the smallest degree d_0 such that for any $d > d_0$ we have $R_d \subseteq R_{\leq\sigma(R)}R$.

A straightforward induction argument gives

$$\beta_k(R) \leq (k-1)\sigma(R) + \eta(R)$$

For abelian groups $H \leq G$ there is a powerful result which combines in a sense the above fact with the reduction lemmata:

$$d_k(G) \leq d_k(H)\exp(G/H) + max\{d(G/H), \eta(G/H) - \exp(G/H) - 1\}$$

This also has a generalization in the framework of the invariant theory of non-abelian groups.

The inductive method and the "contractions"

- for a subgroup $B \leq A$ of an abelian group A consider the natural epimorphism $\phi : A \to A/B$
- for a sequence S over A take a factorization $S = S_0 S_1 .... S_l$ such that $\phi(S_i)$ is a zero-sum sequence over A/B for all $i \geq 1$
- investigate the "contracted" sequence $(\sigma(S_1), ..., \sigma(S_l))$ as a sequence over B (here $\sigma(S_i)$ denotes the sum of a sequence)

This allows to derive information on the zero-sum sequences over A from previous knowledge on the zero-sum sequences over B

We extended this method to a class of non-abelian groups, namely those which have a cyclic subgoup of index 2

What else could be generalized to a non-abelian setting?

- the definition of $s(A)$ and related results, like the Erdos-Ginzburg-Ziv theorem
- the weighted Davenport constant
- the small and the large Davenport constant
- etc. etc.

Thank you for your attention!