# On Primeness in Near-rings of Homogeneous Functions

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October 25, 2014

# 1. Preliminaries

## Definition 1.1

A near-ring is a triple $(N, +, \cdot)$ where

1. $(N, +)$ is a (not necessarily Abelian) group;
2. $(N, \cdot)$ is a semigroup;
3. $(x + y)z = xz + yz$ for all $x, y, z \in N$.

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## Definition 1.2

A subgroup $A$ of $N$ is left (resp. right) invariant if $NA \subseteq A$ (resp. $AN \subseteq A$). If $A$ is both left and right invariant, it is called an invariant subgroup of $N$.

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Let $\mathbb{R}$ and $\mathbb{R}^2$ have their usual topologies. Let

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Then clearly, $f \in M_{\mathbb{R}}(\mathbb{R}^2)$. But $f$ is not continuous at $(1,0)$, so $f \notin N_{\mathbb{R}}(\mathbb{R}^2)$.

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Their action is defined in the natural way, i.e.

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**Theorem 2.1**

*The following statements are equivalent:*

1. *$R$ is a prime ring.*
2. *$N_R(R^2)$ is an equiprime near-ring.*
3. *$N_R(R^2)$ is a 3-prime near-ring.*

*Veldsman (1990), Mogae (2013)*

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## Definition 2.2

Let $A$ and $B$ be subsets of $R$ and $N_R(R^2)$, respectively. Then

$$A^+ = (A^2 : R^2)_N = \{f \in N : f(R^2) \subseteq A^2\} \text{ and}$$
$$B_+ = \left\{x \in R : \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \in B.\right\}$$

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## Theorem 2.3

*Let $R$ be a topological ring, and let $P$ be a right ideal of $R$. Then the following are equivalent:*

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*Let $\begin{bmatrix} u \\ v \end{bmatrix} \in R^2$ and let $a \in N$. Then $a \begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix} = \begin{bmatrix} a\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix}$ and $a \begin{bmatrix} 0 & u \\ 0 & v \end{bmatrix} = \begin{bmatrix} \begin{matrix} 0 \\ 0 \end{matrix} & a\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) \end{bmatrix}$.*

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**Lemma 2.5**

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## Corollary 2.8

*$P$ is a 3-prime ideal of $N$ if and only if it is an equiprime ideal of $N$. Hence $\mathcal{P}_3(N) = \mathcal{P}_e(N)$.*

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*Let $R$ be a commutative PIR. Then $\mathcal{P}_e(N_R(R^2)) = \bigotimes_{i \in I} P_i$, where $P_i = 0$ if $R_i$ is a PID and $P_i = Q_i^+$, where $Q_i$ is the unique nonzero proper prime ideal of $R_i$ in the case that $R_i$ is a special PIR.*

# 3. When $R$ is a Commutative PIR

## Theorem 3.3

*Let $R$ be a commutative PIR. Then the mapping $P \to P^+$ defines a one-to-one correspondence between the set of prime ideals of $R$ and the set of equiprime invariant subgroups of $N_R(R^2)$.*

## Corollary 3.4

*Let $R$ be a special PIR. Then $\mathcal{P}_e(N_R(R^2)) = P^+$, where $P$ is the unique nonzero proper prime ideal of $R$.*

## Corollary 3.5

*Let $R$ be a commutative PIR. Then $\mathcal{P}_e(N_R(R^2)) = \bigotimes_{i \in I} P_i$, where $P_i = 0$ if $R_i$ is a PID and $P_i = Q_i^+$, where $Q_i$ is the unique nonzero proper prime ideal of $R_i$ in the case that $R_i$ is a special PIR.*

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**Corollary 3.6**

*Let $R$ be a commutative PIR. Then* $\mathcal{P}(R)^{+} = \mathcal{P}_e(N_R(R^2))$.

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*Let $R$ be a commutative PIR. Then $\mathcal{P}(R)^+ = \mathcal{P}_e(N_R(R^2))$.*

**Example 3.7**

Let $R = \mathbb{R} \times \mathbb{Z}_4$. Then
$\mathcal{P}_e(N_R(R^2)) = (\mathcal{P}(R))^+ = (\mathcal{P}(\mathbb{R}))^+ \times (\mathcal{P}(\mathbb{Z}_4))^+ = \{0\}^+ \times \{0,2\}^+ = \left\{a \in N_R(R^2) : a(R \times \{0\}) = 0 \text{ and } a(\{0\} \times \mathbb{Z}_4^2) \in \{0,2\}^2\right\}$

# THANK YOU!