On homological dimensions relative to a weakly Wakamatsu tilting module

Driss BENNIS
Rabat - Morocco

joint work with
J. R. Garcia Rozas and L. Oyonarte

Arithmetic and Ideal Theory of Rings and Semigroups

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Outline

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Notations

- R and S will be associative rings with identity.
  All modules will be, unless otherwise specified, unital left R-modules.
- When right R-modules need to be used, they will be denoted as M_R, while in these cases left R-modules (resp, (R, S)-bimodule) will be denoted by ${}_RM$ (resp, ${}_RM_S$).
- We use $Proj(R)$ to denote the class of all projective R-modules.
- The category of all left R-modules will be denoted by R-Mod.
- For an R-module C we use $Add_R(C)$ (resp. $add_R(C)$) to denote the class of all R-modules which are isomorphic to direct summands of direct sums (resp. finite direct sums) of copies of C.
- Given the class $\mathcal{F}$, the class of all modules N such that $\mathrm{Ext}_R^{\geq 1}(F, N) = 0$ for every $F \in \mathcal{F}$ will be denoted by $\mathcal{F}^{\perp}$ (similarly, ${}^{\perp}\mathcal{F} = \{N;\ \mathrm{Ext}_R^{\geq 1}(N, F) = 0\ \forall F \in \mathcal{F}\}$).

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Semidualizing modules

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Araya, Takahashi and Yoshino (2005)

A semidualizing module is an (R, S)-bimodule C satisfying the following properties:

1. ${}_RC$ and C_S both admit a degreewise finite projective resolution in the corresponding module categories (R-Mod and Mod-S).
2. $\mathrm{Ext}_R^{\geq 1}(C, C) = \mathrm{Ext}_S^{\geq 1}(C, C) = 0$.
3. The natural homothety maps $R \to \mathrm{Hom}_S(C, C)$ and $S \to \mathrm{Hom}_R(C, C)$ both are ring isomorphisms.

Semidualizing modules were introduced by Foxby, Golod and Vasconcelos under different names:
- Foxby (1973) called them PG-modules of rank one,
- Vasconcelos (1974) called them Spherical modules, and
- Golod (1984) suitable modules.
- It was Christensen (2001) who used the term "semidualizing".

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Assume that R is a commutative Noetherian ring and C is a semidualizing R-module.

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Definition (Totally C-reflexive module, Golod (1984))

A finitely generated R-module M is called totally C-reflexive if:

- The natural biduality map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism,
- $Ext_R^i(M, C) = 0 = Ext_R^i(Hom_R(M, C), C)$ for $i > 0$.

G_C-projective modules

Assume that R is a commutative Noetherian ring and C is a semidualizing R-module.

Definition (C-Gorenstein projective module, Holm and Jorgensen (2006))

An R-module M is said to be C-Gorenstein projective module if there exists an exact sequence of R-modules

$$\mathbf{X} = \ \cdots \to P_1 \to P_0 \to C \otimes Q^0 \to C \otimes Q^1 \to \cdots$$

where every P_i and $Q^i s$ are projective for every $i \in \mathbb{N}$, $M \cong \mathrm{Im}(P_0 \to C \otimes Q^0)$, and such that $\mathrm{Hom}_R(-, C \otimes Q)$ leaves the sequence $\mathbf{X}$ exact whenever Q is a projective module.

G_C-projective modules

- White (2010) extended the study of C-Gorenstein projective modules to not necessarily Noetherian rings, where she used the term "G_C-projective" instead of "C-Gorenstein projective".
- For any arbitrary ring R and a semidualizing (R, S)-bimodule C, Geng and Ding (2011) showed that the set of all R-modules of the form $C \otimes_R P$, with P is projective, coincide with the set $Add_S(C)$.

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G_C-projective modules

Definition (G_C-projective module; Liu, Huang and Xu (2013))

For any arbitrary ring R and a semidualizing (R, S)-bimodule C, an R-module M is said to be G_C-projective if there exists an exact sequence of R-modules

$$\mathbf{X} = \ \cdots \to P_1 \to P_0 \to A^0 \to A^1 \to \cdots$$

where the $P_i's$ are all projective, $A^i \in Add_R(C)$ for every $i \in \mathbb{N}$, $M \cong \mathrm{Im}(P_0 \to A^0)$, and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{X}$ exact whenever $Q \in Add_R(C)$.
We say that the sequence $\mathbf{X}$ is a $\mathrm{Hom}_R(-, Add_R(C))$-exact.
We use $G_CP(R)$ to denote the class of all G_C-projective R-modules.

Principal results

- $Proj(R) \subseteq G_CP(R)$.
- $Add_R(C) \subseteq G_CP(R)$.
- The class $G_CP(R)$ is projectively resolving; that is, for every short exact sequence of R-modules $0 \to N \to M \to L \to 0$, where $L \in G_CP(R)$, we have: $M \in G_CP(R)$ **if and only if** $N \in G_CP(R)$.
- For every family of R-modules $(M_i)_{i \in I}$, the sum $\oplus G_i \in G_CP(R)$ if and only if every $M_i \in G_CP(R)$.

Question.

Is the condition on C to be semidualizing necessary so that the relative homological dimensions preserve their properties?

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G_C-projective modules associated to a Σ-self-orthogonal module C

Let C be an R-module. The class $G_CP(R)$ is closed under direct sums and extensions.

Let C be an R-module. The following conditions are equivalent:

1. $C \in G_CP(R)$.
2. C is Σ-self-orthogonal, that is $\mathrm{Ext}_R^{\geq 1}(C, C^{(I)}) = 0$ for every set I.
3. $Add_R(C) \subseteq Add_R(C)^{\perp}$.
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Let C be a Σ-self-orthogonal R-module. Then, the class $G_CP(R)$ is projectively resolving.
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Definition

An R-module C is weakly Wakamatsu tilting if it has the following two properties:

1. C is Σ-self-orthogonal. (that is, $C^{(I)} \in Add_R(C)^{\perp}$ for every set I).
2. There exists a $\mathrm{Hom}_R(-, Add_R(C))$-exact exact sequence of R-modules

$$\mathbf{X}: \ 0 \longrightarrow R \xrightarrow{f_0} C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} \cdots$$

where, for every $i \in \mathbb{N}$, $C_i \in Add_R(C)$.

Mantese and Reiten (2004)

An R-module C is Wakamatsu tilting if:

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where, for every $i \in \mathbb{N}$, $C_i \in add_R(C)$.

(Wakamatsu 2004) If M is a left R-module and $S = \mathrm{End}_R(M)$ then ${}_R M_S$ is semidualizing if and only if ${}_R M$ is Wakamatsu tilting, if and only if M_S is Wakamatsu tilting with $R = \mathrm{End}_S(M)$.

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(Wakamatsu 2004) If M is a left R-module and $S = \mathrm{End}_R(M)$ then ${}_RM_S$ is semidualizing if and only if ${}_RM$ is Wakamatsu tilting, if and only if M_S is Wakamatsu tilting with $R = \mathrm{End}_S(M)$.

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Let C be a Σ-self-orthogonal R-module. Then, the following conditions are equivalent:

1. C is weakly Wakamatsu tilting.
2. $R \in G_C P(R)$.
3. $Proj(R) \subseteq G_C P(R)$.

G_C-projective modules associated to a weakly Wakamatsu tilting module C

Let C be a Σ-self-orthogonal R-module. Then the following assertions are equivalent.
1) $M \in G_CP(R)$ if and only if there exists a $\mathrm{Hom}_R(-, Add_R(C))$-exact exact sequence $\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$ in R-Mod with $C_i, C^i \in Add_R(C)$ and $M \cong \mathrm{Im}(C_0 \to C^0)$.
2) C is weakly Wakamatsu tilting.

Interaction with Bass class

Interaction with Bass class

Let C be an R-module.
Denote by $C - GP(R)$ the class of all left R-modules M such that there exists $\mathrm{Hom}_R(Add_R(C), -)$-exact and $\mathrm{Hom}_R(-, Add_R(C))$-exact exact sequence of R-modules $\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$ with $C_i, C^i \in Add_R(C)$ and $M \cong \mathrm{Im}(C_0 \to C^0)$.

Definition

Let C be an R-module and consider $S = \mathrm{End}_R(C)$.
The Bass classe $\mathcal{B}_C(R)$ consists of all left R-modules N satisfying:

B1) $\mathrm{Ext}_R^{\geq 1}(C, N) = 0$,
B2) $\mathrm{Tor}^S_{\geq 1}(C, \mathrm{Hom}_R(C, N)) = 0$,
B3) the canonical map $\nu_N : C \otimes_S \mathrm{Hom}_R(C, N) \to N$ is an isomorphism of R-modules.

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Interaction with Bass class

Geng and Ding (2011)

If R is commutative and C is semidualizing, then $C - GP(R) = G_CP(R) \cap \mathcal{B}_C(R)$.

Interaction with Bass class

Theorem

Let C be an R-module and consider $S = \mathrm{End}_R(C)$.
If C_S *is* $\otimes$*-faithful (i.e. it satisfies:* $N \otimes_R C = 0 \Rightarrow N = 0$*), then the following statements are equivalent.*
i) $Add_R(C) \subseteq \mathcal{B}_C(R)$.
ii) $C - GP(R) = G_C P(R) \cap \mathcal{B}_C(R)$.
iii) ${}_R C$ *is* Σ*-self-orthogonal and self-small (i.e. it satisfies:* $\mathrm{Hom}_R(M, M^{(I)}) \cong \mathrm{Hom}_R(M, M)^{(I)}$ *for every set I).*

P_C-projective dimension

Definition

A module M is said to have finite P_C-projective dimension, $\mathrm{P_C-pd}(M) < \infty$, if there is an exact sequence

$$0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0 \text{ with } A_i \in Add_R(C) \text{ for}$$

every $i \in \{0, ..., n\}$.

$\mathrm{P_C-pd}(M)$ is the least nonnegative integer n for which such a sequence exists, and if there is no such n then $\mathrm{P_C-pd}(M) = \infty$.

- Takahashi and White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010).

Definition

Let T be a class of modules. By a left T-resolution of a module M, we mean a $Hom(T, -)$-exact complex $\cdots \to T_1 \to T_0 \to M \to 0$ (not necessarily exact) with each $T_i \in T$.

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$$0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0 \text{ with } A_i \in Add_R(C) \text{ for}$$

every $i \in \{0, ..., n\}$.

$\mathrm{P_C-pd}(M)$ is the least nonnegative integer n for which such a sequence exists, and if there is no such n then $\mathrm{P_C-pd}(M) = \infty$.

- Takahashi and White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010).

Definition

Let T be a class of modules. By a left T-resolution of a module M, we mean a $Hom(T, -)$-exact complex $\cdots \to T_1 \to T_0 \to M \to 0$ (not necessarily exact) with each $T_i \in T$.

P_C-projective dimension

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P_C-projective dimension

Theorem

Let C be a $\sum$-self-orthogonal R-module. For an R-module M and an integer number $n \geq 0$ the following assertions are equivalent:

1. $\mathrm{P_C}-\mathrm{pd}(M) \leq n$.

2. There is an exact left $Add_R(C)$-resolution of M of length n.

3. For every left $Add_R(C)$-resolution

$$\cdots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} M \xrightarrow{f_{-1}} 0$$

of M, $\ker(f_{n-1}) \in Add_R(C)$ and the left $Add_R(C)$-resolution $0 \to \ker(f_{n-1}) \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 \to M \to 0$ is exact.

P_C-projective dimension

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- A left R-module M is said to be Hom-faithful if: $\mathrm{Hom}_R(M, N) = 0 \Rightarrow N = 0$.
- Let T be a class of R-modules. Then for an R-module M, a homomorphism $f : C \to M$ where $C \in T$ is called a T-precover of M if, for every $C' \in T$, $f^* : \mathrm{Hom}(C', C) \to \mathrm{Hom}(C', M)$ is surjective.

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P_C-projective dimension

Theorem

Let C be a $\sum$-self-orthogonal R-module. The following assertions are equivalent:

1. C *is* Hom*-faithful.*
2. *If* $\phi : I \to M$ *is an* $Add_R(C)$*-precover with* $K = \ker(\phi) \in Add_R(C)^{\perp}$*, then* ϕ *is surjective and* $M \in Add_R(C)^{\perp}$*.*
3. *If* $\cdots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} M \longrightarrow 0$ *is a left* $Add_R(C)$*-resolution of* M *with* $\ker(f_n) \in Add_R(C)^{\perp}$*, then the sequence* $0 \to \ker(f_n) \to \cdots \to Y_1 \to Y_0 \to M \to 0$ *is exact.*
4. *Every monic* $Add_R(C)$*-precover* $\phi : I \to M$ *of an* R*-module* M *is an isomorphism.*
5. *For every* R*-module* M *and every* $n \in \mathbb{N}$, $\mathrm{P_C{-}pd}(M) \leq n$ *if and only if* M *admits a left* $Add_R(C)$*-resolution of length* n*.*

P_C-projective dimension

Theorem

Let C be a $\sum$-self-orthogonal R-module. The following assertions are equivalent:

1. *C is Hom-faithful.*
2. *If $\phi : I \to M$ is an $Add_R(C)$-precover with $K = \ker(\phi) \in Add_R(C)^{\perp}$, then ϕ is surjective and $M \in Add_R(C)^{\perp}$.*
3. *If $\cdots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} M \longrightarrow 0$ is a left $Add_R(C)$-resolution of M with $\ker(f_n) \in Add_R(C)^{\perp}$, then the sequence $0 \to \ker(f_n) \to \cdots \to Y_1 \to Y_0 \to M \to 0$ is exact.*
4. *Every monic $Add_R(C)$-precover $\phi : I \to M$ of an R-module M is an isomorphism.*
5. *For every R-module M and every $n \in \mathbb{N}$, $\mathrm{P_C{-}pd}(M) \leq n$ if and only if M admits a left $Add_R(C)$-resolution of length n.*

P_C-projective dimension

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Theorem

Let C be a weakly Wakamatsu tilting R-module. For every R-module M, $\mathrm{G_C-pd}(M) \leq \mathrm{P_C-pd}(M)$*, such that the equality* $\mathrm{P_C-pd}(M) = \mathrm{G_C-pd}(M)$ *holds true whenever M has a finite* P_C*-projective dimension.*

P_C-projective dimension

Example (Example of a module M which has finite G_C-projective dimension but infinite P_C-projective dimension)

Consider any left noetherian ring R, and a Gorenstein injective left R-module M which is not injective. If

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

is the complete injective resolution associated to M, that is, $M = \ker(E^0 \to E^1)$, take $C = (\oplus_{i\geq 0} E^i) \oplus (\oplus_{i\geq 1} E^i(R))$ where $0 \to R \to E^1(R) \to E^2(R) \to \cdots$ is an injective resolution of R. Then C is weakly Wakamatsu tilting and M is G_C-projective. However $\mathrm{P_C-pd}(M) = \infty$.

Thank you for your time