New Developments for the Plus-Minus Davenport Constant

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- Benjamin Wright, Providence College

G is an additive finite abelian group.

Consider $\mathscr{F}(G)$, the monoid of finite sequences $S = g_1 g_2 \dots g_n$ over G, under the operation of sequence concatenation.

We have a special submonoid called the **block monoid**:

$$\mathscr{B}(G) = \{S \in \mathscr{F}(G) \,|\, g_1 + g_2 + \ldots + g_n = 0\}$$

which consists of all the **zero-sum sequences** over G.

The block monoid plays a key role in the factorization theory of algebraic number rings (and Dedekind domains and Krull monoids), since such rings D have a transfer homomorphism $D \to \mathscr{B}(C(D))$, where $C(D)$ is the divisor class group.

We can factor zero-sum sequences in $\mathscr{B}(G)$ as a product of zero-sum subsequences.

$$S = S_1 S_2 \cdots S_k$$

The atoms of $\mathscr{B}(G)$ are the **minimal zero-sum sequences**, namely, zero-sum sequences S with no proper zero-sum subsequence.

The **Davenport constant** $D(G)$ is the longest length of a minimal zero-sum sequence over G.

Equivalent definitions of the Davenport constant $D(G)$:

A **zero-sum-free sequence** is a sequence $S \in \mathscr{F}(G)$ which has no subsequence that is zero sum.

• $D(G)$ is the least n such that there are no zero-sum-free sequences of length n.

• $D(G)$ is the least n such that for every $S = g_1 \cdots g_n \in \mathscr{F}(G)$, there exist $a_1, \ldots, a_n \in \{0, 1\}$ not all zero such that

$$\sum_{i=1}^{n} a_i g_i = 0$$

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The **weighted Davenport constant** $D_A(G)$ with weight set $A \subseteq \mathbb{Z}$ is the least n such that for every $S = g_1 \cdots g_n \in \mathscr{F}(G)$, there exist $a_1, \ldots, a_n \in A$ not all zero such that

$$\sum_{i=1}^{n} a_i g_i = 0$$

We can also say $S \in \mathscr{F}(G)$ is **weighted zero-sum-free (wzsf)** if whenever $a_1, \ldots, a_n \in A$ satisfy

$$\sum_{i=1}^{n} a_i g_i = 0$$

then $a_1 = \ldots = a_n = 0$. Clearly, $D_A(G)$ is the least n such that there are no wzsf of length n.

As long as $1 \in A$ (which we require), $D_A(G) \leq D_{\{1\}}(G) = D(G)$

Study began only in 2006 by Adhikari and his coauthors.

- S. D. Adhikari, Y. G. Chen, J. B. Friedlander, S. V. Konyagin and F. Pappalardi, *Contributions to zero-sum problems*, Discrete Math. 306 (2006) 110.
- S. D. Adhikari and P. Rath, *Davenport constant with weights and some related questions*, Integers 6 (2006) A30, 6 pp.

We care about $A = \{-1, 0, 1\}$ and denote $D_A(G)$ by $D_{\pm}(G)$, the **plus-minus Davenport constant**.

$D_{\pm}(G)$ plays a role in:

- creating dissociated sets, used in Fourier arguments and integer lattices (see Tao, Vu *Additive Combinatorics*)
- norms of principal ideals in quadratic algebraic number fields (Halter-Koch)
- factorization problems when you refine the definition of "associates" (B., Chris Mooney)

Theorem (Adhikari, Grynkiewicz, Sun 2012)

If $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ *for* $n_1 | n_2 | \dots | n_r$*, then*

$$1 + \sum_{i=1}^{r} \lfloor \log_2(n_i) \rfloor \leq D_{\pm}(G) \leq 1 + \left\lfloor \sum_{i=1}^{r} \log_2(n_i) \right\rfloor = 1 + \lfloor \log_2(|G|) \rfloor$$

Marchan, Ordaz, and Schmid noted the hypothesis "or $n_1 | n_2 | \dots | n_r$" was superfluous and it's advantageous sometimes to use other expressions for G because you get tighter bounds.

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Remarks on the AGS bounds

$$1 + \sum_{i=1}^{r} \lfloor \log_2(n_i) \rfloor \leq D_{\pm}(G) \leq 1 + \left\lfloor \sum_{i=1}^{r} \log_2(n_i) \right\rfloor$$

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- Corollary: $D_{\pm}(C_n) = \lfloor \log_2(n) \rfloor + 1$
- Corollary: For any 2-group G, $D_{\pm}(G) = \log_2(|G|) + 1$.
- Corollary: For any 2-group G and any $n \geq 1$, $D_{\pm}(G \oplus C_n) = \log_2(|G|) + \lfloor \log_2(n) \rfloor + 1$.

Other values:

Theorem (Thangadurai 2007)

For all $r \geq 1$, $D_{\pm}(C_3^r) = r + 1$.

Marchan, Ordaz and Schmid (2014) developed some general lemmas to try to attack C_n^2, $C_n + C_m$, $C_n + C_m + C_q$, and C_p^r, but no general formula.

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Determined $D_{\pm}(G)$ for all groups of order ≤ 100 except $G = C_5 + C_{15}$. In each case, $D_{\pm}(G)$ equalled the upper or lower bound of AGS (usually the upper).

Question: Are there finite abelian groups with

$$1 + \sum_{i=1}^{r} \lfloor \log_2(n_i) \rfloor < D_{\pm}(G) < 1 + \left\lfloor \sum_{i=1}^{r} \log_2(n_i) \right\rfloor$$

(using the best most favorable representation of G) ?

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Known values:

$D_\pm(C_3 + C_2) = D_\pm(C_6) = 3 = lb = ub$
$D_\pm(C_3^2 + C_2) = 5 = ub$
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First unknown: $D_\pm(C_3^4 + C_2)$.

Thm $D_\pm(C_3^4 + C_2) = 7$ and $D_\pm(C_3^5 + C_2) = 8$

$$lb = 6 < D_\pm(C_3^4 + C_2) < ub = 8$$
$$lb = 7 < D_\pm(C_3^5 + C_2) < ub = 9$$

MOS question has POSITIVE answer (as expected).

Lemma

For all $r \geq 2$, $D_{\pm}(C_3^r + C_2) \geq r + 3$.

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Proof: Construct a wzsf sequence of length $r + 2$. For r even, take

$$\begin{aligned}
s_1 &= (1, 0, 0, 0, \ldots, 0, 0, 1)\\
s_2 &= (0, 1, 0, 0, \ldots, 0, 0, 1)\\
s_3 &= (0, 0, 1, 0, \ldots, 0, 0, 1)\\
&\vdots \qquad\qquad \vdots\\
s_r &= (0, 0, 0, 0, \ldots, 0, 1, 1)\\
s_{r+1} &= (1, 1, 1, 1, \ldots, 1, 1, 1)\\
s_{r+2} &= (1, 2, 0, 0, \ldots, 0, 0, 1)
\end{aligned}$$

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&\vdots \\
s_r &= (0,0,0,0,\ldots,0,1,1) \\
s_{r+1} &= (1,1,1,1,\ldots,1,1,0) \\
s_{r+2} &= (1,2,0,0,\ldots,0,0,1)
\end{aligned}$$

Conjecture for the entire summer was: For all $r \geq 2$, $D_{\pm}(C_3^r + C_2) = r + 3$.

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Conjecture is false!

Theorem

For $2 \leq r \leq 9$, $D_{\pm}(C_3^r + C_2) = r + 3$.
BUT, $D_{\pm}(C_3^{10} + C_2) = 14 = r + 4$

This is a group of order 118,098. No hope for brute force search.

The wzfs of length 13 that we found in $C_3^{10} + C_2$ is

$$
\begin{aligned}
s_1 &= (\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ )\\
s_2 &= (\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ )\\
s_3 &= (\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ )\\
s_4 &= (\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ )\\
s_5 &= (\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ )\\
s_6 &= (\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ )\\
s_7 &= (\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ )\\
s_8 &= (\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ )\\
s_9 &= (\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ )\\
s_{10} &= (\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ )\\
s_{11} &= (\ 1\ 1\ 1\ 2\ 1\ 1\ 0\ 1\ 2\ 0\ 1\ )\\
s_{12} &= (\ 1\ 1\ 2\ 1\ 1\ 0\ 1\ 2\ 0\ 1\ 1\ )\\
s_{13} &= (\ 1\ 2\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 2\ 1\ )
\end{aligned}
$$

$D_\pm(C_3^1 + C_2) = 3 = r + 2$
$D_\pm(C_3^2 + C_2) = 5 = r + 3$
$D_\pm(C_3^3 + C_2) = 6 = r + 3$
$D_\pm(C_3^4 + C_2) = 7 = r + 3$
$D_\pm(C_3^5 + C_2) = 8 = r + 3$
$D_\pm(C_3^6 + C_2) = 9 = r + 3$
$D_\pm(C_3^7 + C_2) = 10 = r + 3$
$D_\pm(C_3^8 + C_2) = 11 = r + 3$
$D_\pm(C_3^9 + C_2) = 12 = r + 3$
$D_\pm(C_3^{10} + C_2) = 14 = r + 4$
We can also now prove

$$D_\pm(C_3^r + C_2) + 1 \le D_\pm(C_3^{r+1} + C_2) \le D_\pm(C_3^r + C_2) + 2$$

so jumps of 2 are the worst. When do they happen?

We think the actual formula is given by: if

$$\frac{3^k - 1}{2} - k \leq r < \frac{3^{k+1} - 1}{2} - (k+1)$$

then $D_\pm(C_3^r + C_2) = r + k + 1$.

Higher exponents? $D_{\pm}(C_3^r + C_2^s)$?

Proposition

$D_{\pm}(G + H) \geq D_{\pm}(G) + D_{\pm}(H) - 1$

Sample use:

Proposition

$D_{\pm}(C_3^{2r} + C_2^{r+s}) \geq 4r + s + 1$

Good enough to get the precise value of small groups:

Corollary

$D_{\pm}(C_3^{2r} + C_2^{r+s}) = 4r + s + 1$ *for all* $1 \leq r \leq 5$ *and all* $s \geq 0$.

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