On the dot product graph of a commutative ring

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Definition

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. Recently, there has been considerable attention in the literature to associating graphs with algebraic structures. Probably the most attention has been to the zero-divisor graph $\Gamma(R)$ for a commutative ring R in the sense of Beck-Anderson-Livingston. The set of vertices of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $xy = 0$.

Definition

Let A be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and let $R = A \times A \times \cdots \times A$ (n times). Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in R$. Then the dot product $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in A$. In this talk, we introduce the total dot product graph of R to be the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, ..., 0)\}$, and two distinct vertices x and y are adjacent if and only if $x \cdot y = 0 \in A$. Let $Z(R)$ denote the set of all zero-divisors of R. Then the zero-divisor dot product graph of R is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, ..., 0)\}$. It follows that each edge (path) of the classical zero-divisor graph $\Gamma(R)$ is an edge (path) of $ZD(R)$. We observe that if $n = 1$, then $TD(R)$ is a disconnected graph, where $ZD(R)$ is identical to $\Gamma(R)$ in the sense of Beck-Anderson-Livingston, and hence it is connected.

Definition

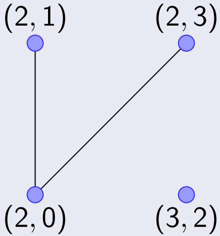
We recall some definitions. Let Γ *be a (undirected) graph. We say that* Γ *is connected if there is a walk (path) between any two distinct vertices. For vertices* x *and* y *of* Γ*, we define* $d(x, y)$ *to be the length of a shortest path from* x *to* y *(*$d(x, x) = 0$ *and* $d(x, y) = \infty$ *if there is no path). Then the diameter of* Γ *is* $diam(\Gamma) = sup\{\ d(x, y) \mid x \text{ and } y \text{ are vertices of } \Gamma\ \}$*. The girth of* Γ*, denoted by* $gr(\Gamma)$*, is the length of a shortest cycle in* Γ *(*$gr(\Gamma) = \infty$ *if* Γ *contains no cycles). A graph* Γ *is complete if any two distinct vertices are adjacent.*

Theorem

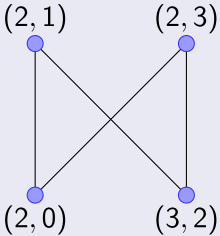
Let A be an integral domain and $R = A \times A$. Then $TD(R)$ is disconnected and $ZD(R) = \Gamma(R)$ is connected. In particular, if A is ring-isomorphic to $\mathbb{Z}_2$, then $ZD(R)$ is complete (i.e., $diam(ZD(R)) = 1$) and $gr(ZD(R)) = \infty$. If A is not ring-isomorphic to $\mathbb{Z}_2$, then $diam(ZD(R)) = 2$ and $gr(ZD(R)) = 4$.

Example

Let $A = Z_4$, $R = A \times A = Z_4 \times Z_4$. *Part of the* $\Gamma(R)$



Let $A = Z_4$, $R = A \times A = Z_4 \times Z_4$. *Part of the* $ZD(R)$



Theorem

Let $2 \leq n < \infty$, A be a commutative ring with $1 \neq 0$, and $R = A \times A \times \cdots \times A$ (n times). Then $ZD(R) = \Gamma(R)$ if and only if either $n = 2$ and A is an integral domain or R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem

Let A be a commutative ring with $1 \neq 0$ that is not an integral domain, and let $R = A \times A$. Then the following statements hold.

1. *$TD(R)$ is connected and $diam(TD(R)) = 3$.*
2. *$ZD(R)$ is connected, $ZD(R) \neq \Gamma(R)$, and $diam(ZD(R)) = 3$.*
3. *$gr(ZD(R)) = gr(TD(R)) = 3$.*

Theorem

Let A be a commutative ring with $1 \neq 0$, $3 \leq n < \infty$, and let $R = A \times A \times \cdots \times A$ (n times). Then $TD(R)$ is connected and $diam(TD(R)) = 2$.

Theorem

Let A be a commutative ring with $1 \neq 0$. Then the following statements hold.

1. *If A is an integral domain and $R = A \times A \times A$, then $ZD(R)$ is connected ($ZD(R) \neq \Gamma(R)$) and $diam(ZD(R)) = 3$.*
2. *If A is not an integral domain and $R = A \times A \times A$, then $ZD(R)$ is connected ($ZD(R) \neq \Gamma(R)$) and $diam(ZD(R)) = 2$.*
3. *If $4 \leq n < \infty$ and $R = A \times A \times \cdots \times A$ (n times), then $ZD(R)$ is connected ($ZD(R) \neq \Gamma(R)$) and $diam(ZD(R)) = 2$.*

Theorem

Let A be a commutative ring with $1 \neq 0$, $3 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then $gr(ZD(R)) = gr(TD(R)) = 3$.

Corollary

Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent.

1. $gr(ZD(R)) = 3$.
2. $gr(TD(R)) = 3$.
3. *A is not an integral domain and $n = 2$ or $n \geq 3$.*

Corollary

Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent.

1. $gr(ZD(R)) = \infty$.
2. *A is ring-isomorphic to $\mathbb{Z}_2$ and $n = 2$.*
3. $diam(ZD(R)) = 1$.

Corollary

Let A be a commutative ring with $1 \neq 0$ such that A is not ring-isomorphic to $\mathbb{Z}_2$, $0 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent.

1. $gr(ZD(R)) = 4$.
2. $ZD(R) = \Gamma(R)$.
3. $TD(R)$ *is disconnected.*
4. $n = 2$ *and A is an integral domain.*

Corollary

Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent.

1. $diam(ZD(R)) = 3$.
2. *Either A is not an integral domain and $n = 2$ or A is an integral domain and $n = 3$.*

Corollary

Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent:

1. $diam(ZD(R)) = 2$.
2. *Either A is an integral domain that is not ring-isomorphic to $\mathbb{Z}_2$ and $n = 2$, A is is not an integral domain and $n = 3$, or $n \geq 4$.*

Corollary

Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then $diam(TD(R)) = 3$ if and only if A is not an integral domain and $n = 2$.

Corollary

Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent.

1. $diam(TD(R)) = 2$.
2. $TD(R)$ *is connected and* $n \geq 3$.
3. $n \geq 3$.

Corollary

Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then $diam(TD(R)) = diam(ZD(R)) = 3$ if and only if A is not an integral domain and $n = 2$.

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