Graph-Laplacian minimisation for surface smoothing in 3D finite element tetrahedral meshes

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Simulations of the heart: Requires tetrahedral mesh.

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Mesh quality: Very flat tetrahedra cause errors in simulation.

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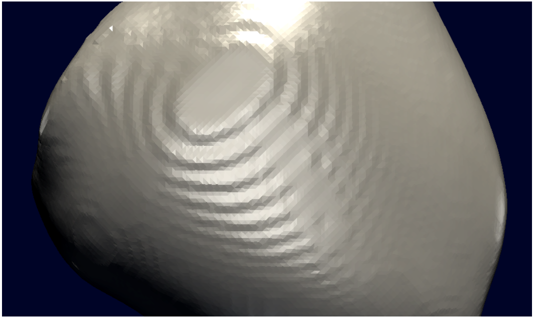

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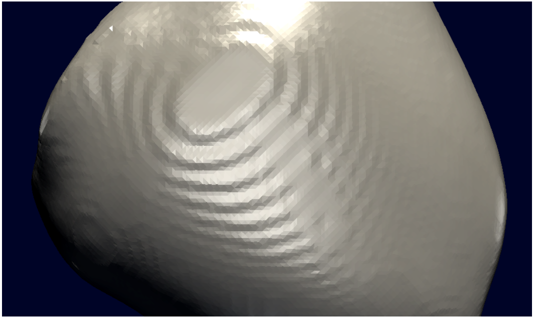

Initial setting: Triangulation is given. Coordinates of vertices, edge information and masking of surface points.

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Primal-dual algorithm: Solution of resulting optimisation problem.

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2 The optimisation problem

3 Numerical solution

4 Discussion of results

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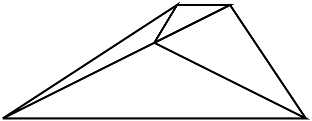

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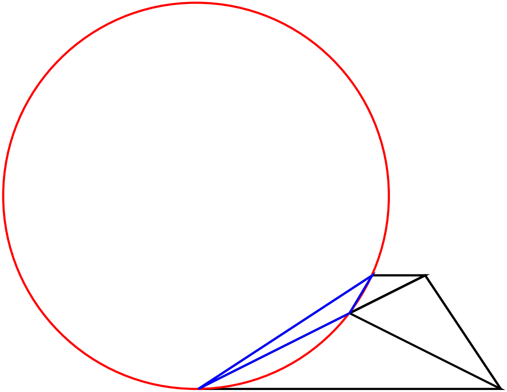

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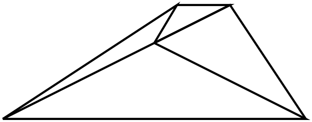

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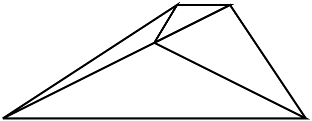

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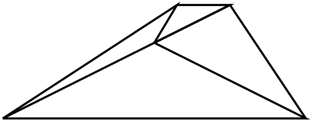
Initial mesh: Assumed to have sufficiently high mesh quality.



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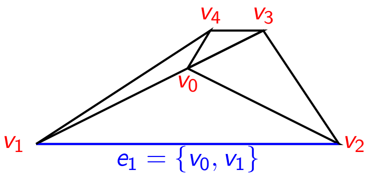

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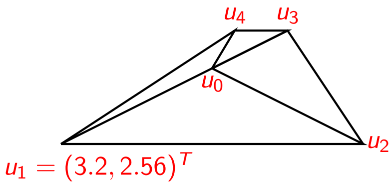

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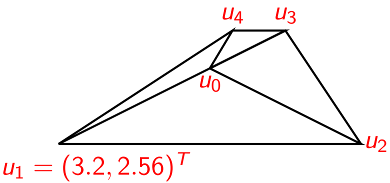
$$(\hat{\Delta})_{i,j} := \begin{cases} \mathrm{Deg}(v_i) & \text{if } i = j, \\ -1 & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{else,} \end{cases}$$



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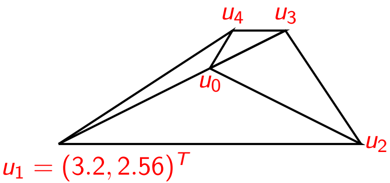

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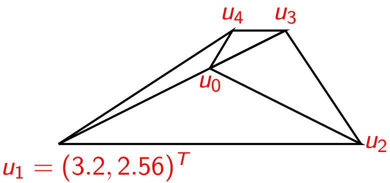
- Mean value and curvature: $(\Delta u)_0 = 4u_0 - u_1 - u_2 - u_3 - u_4$.



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Local, vertex wise gradient descent

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$$u^+ \in \operatorname*{argmin}_{u \in U} \frac{1}{2}\|\Delta u\|_2^2. \tag{2}$$

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- Fit to data.
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- Convex, vertex wise independent representation of Ω.

Update surface points by solving

$$(3) \qquad u^{+} \in \operatorname*{argmin}_{u \in U} \frac{1}{2}\|\Delta u\|_2^2, \quad \text{subject to } u \in \Omega,$$

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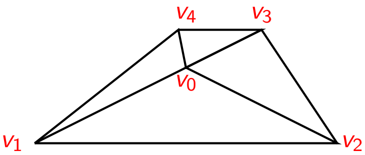

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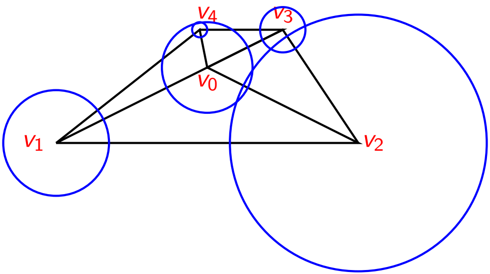

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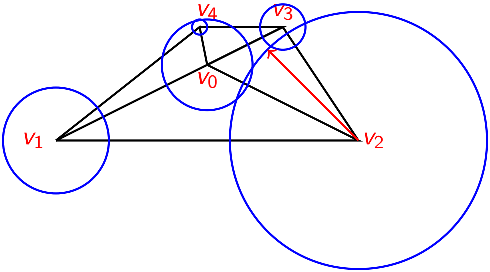

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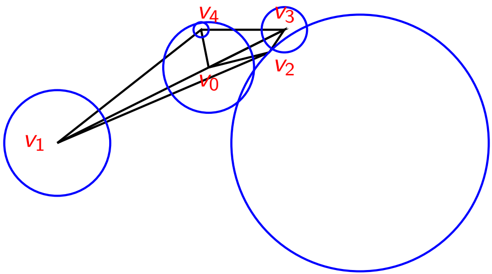

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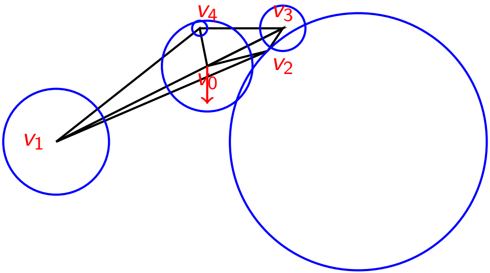

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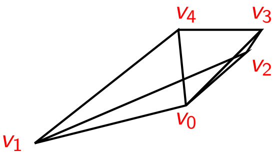

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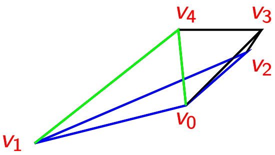

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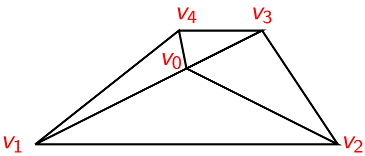

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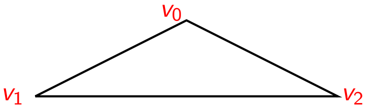

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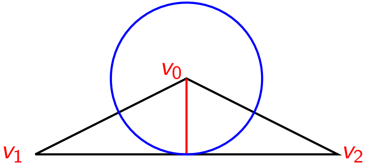

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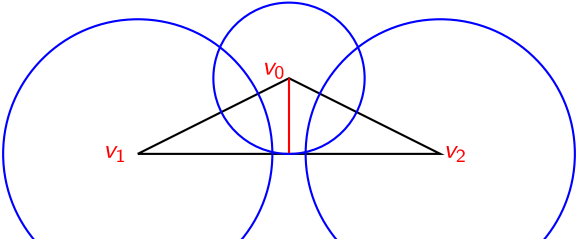

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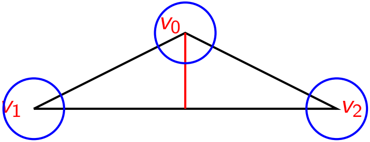

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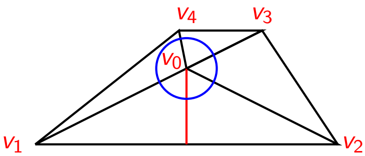

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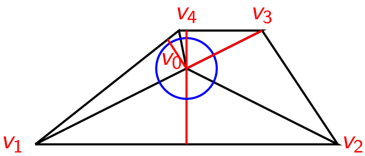

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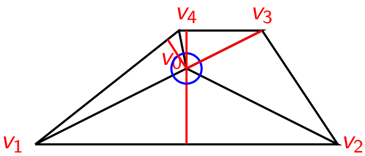
$$r_i = \alpha \hat{h}_i \quad \text{for } 0 < \alpha < 1/2 \text{ with } \hat{h}_i = \min\{h_T\colon v_i \in T\},$$

$$\Omega = \{u \in U\colon \|u_i - u_{0i}\| \leq r_i \text{ for } i = 1, \dots, N\}. \tag{4}$$



Adaptive constraints:

- Incorporating topology into constraints.
- Maintain high mesh quality.

Considerations:

- Heights determine maximal range of movement.
- Synchronous movement, further reduction of radius.
- v_0 contained in many tetrahedra.

$$h_T = \min\{h\colon h \text{ height of } T\} \text{ for tetrahedron } T,$$

$$r_i = \alpha \hat{h}_i \quad \text{for } 0 < \alpha < 1/2 \text{ with } \hat{h}_i = \min\{h_T\colon v_i \in T\},$$

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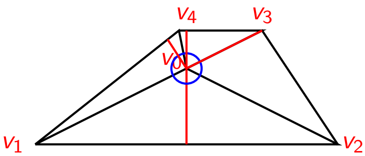
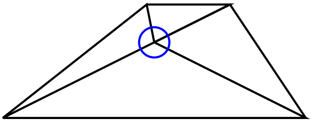
- Note that also interior points are considered in the construction of constraints.



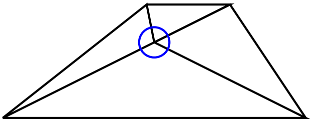
Restrictive constraints:

Restrictive constraints: Little movement allowed.



Restrictive constraints: Little movement allowed.

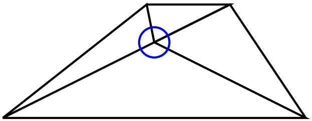
Reiteration of method: Start again with updated constraints.



Restrictive constraints: Little movement allowed.

Reiteration of method: Start again with updated constraints.

- More flexibility and movement.



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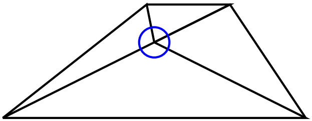
- Constraints ensure mesh quality.



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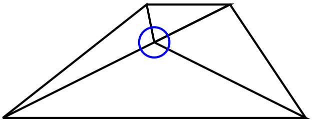
- Many iterations affect quality.



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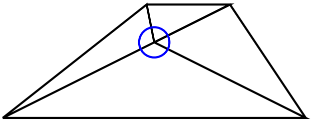
- 2-5 outer iteration.



Restrictive constraints: Little movement allowed.

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- More flexibility and movement.
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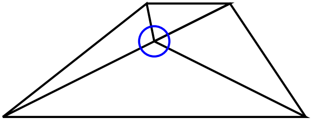
- Many iterations affect quality.
- 2-5 outer iteration.
- Improves results.



Outline

1 Preliminaries

2 The optimisation problem

3 Numerical solution

4 Discussion of results

Primal-dual algorithm: Global solution via iterative method.

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General saddle-point problem for linear A and G:

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Specific saddle-point problem for $A = \Delta$ and $G = \frac{1}{2}\|\cdot\|_2^2$

$$\min_{u\in U}\max_{w\in U} \quad \langle w, \Delta u\rangle - \frac{1}{2}\|w\|_2^2 + I_\Omega(u). \tag{5}$$

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1. *The original optimisation problem* (3), $\min_{u\in\Omega} F(u) = \frac{1}{2}\|\Delta u\|_2^2$, *has a solution which however is not unique.*
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3. *Further, for any saddle-point* (u^+, w^+) *of* (5), u^+ *is a solution of the original minimisation problem* (3).

Specific iteration directives:

Specific iteration directives: Simplicity of constraints leads to

$$(6) \qquad \left\{ \vphantom{\begin{matrix} a \\ b \\ c \\ d \end{matrix}} \right.$$

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$$
\begin{cases}
w \leftarrow \frac{(w+\sigma\Delta\bar{u})}{(1+\sigma)} \\
\bar{u} \leftarrow P_\Omega(u - \tau\Delta w) \\
u \leftarrow 2\bar{u} - u \\
(u, \bar{u}) \leftarrow (\bar{u}, u)
\end{cases} \tag{6}
$$

- Matrix-vector multiplication.
- With $P_\Omega(u)$ pointwise projection.
- Simple operations, vectorisation, fast execution.

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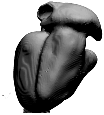
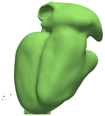
Visual results Example 1:



original



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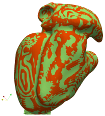

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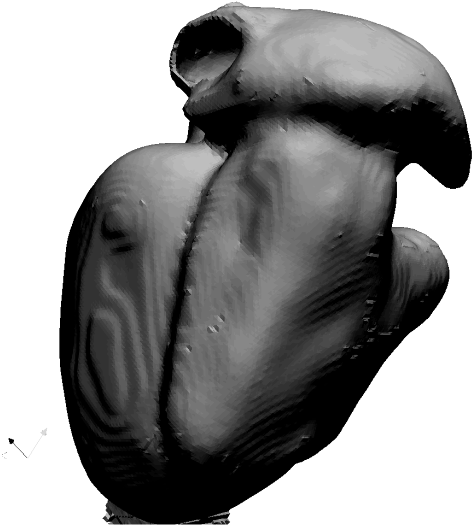
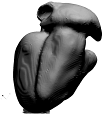


Oscillations:

Visual results Example 1:



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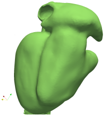

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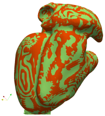

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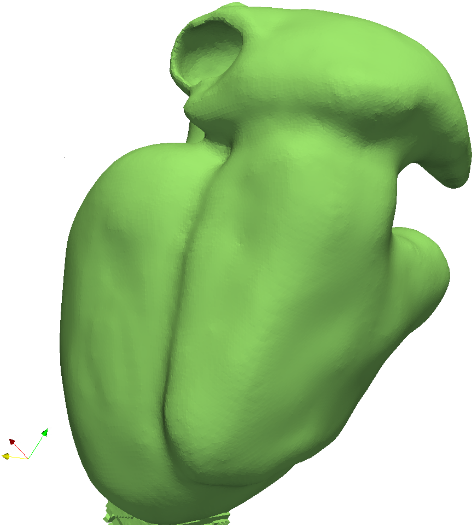

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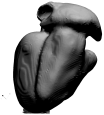

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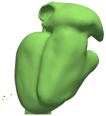

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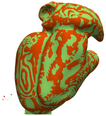

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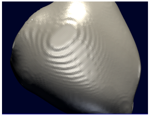

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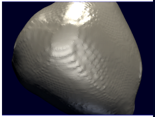

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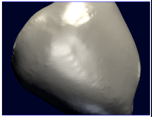

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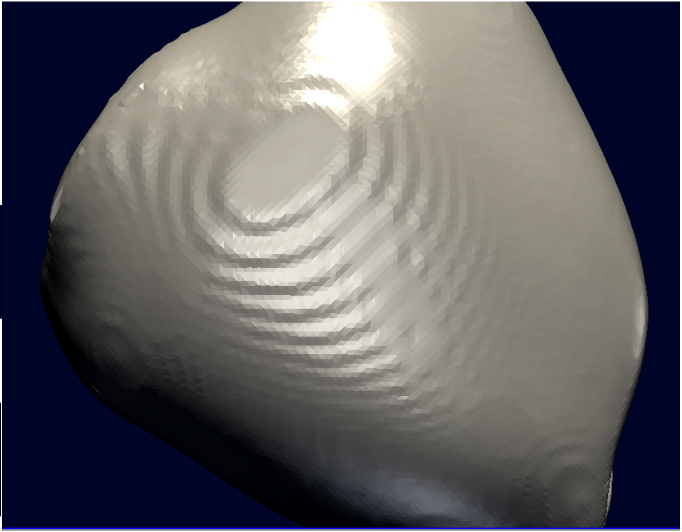

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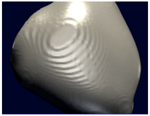

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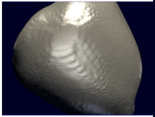

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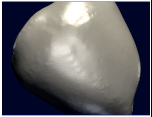

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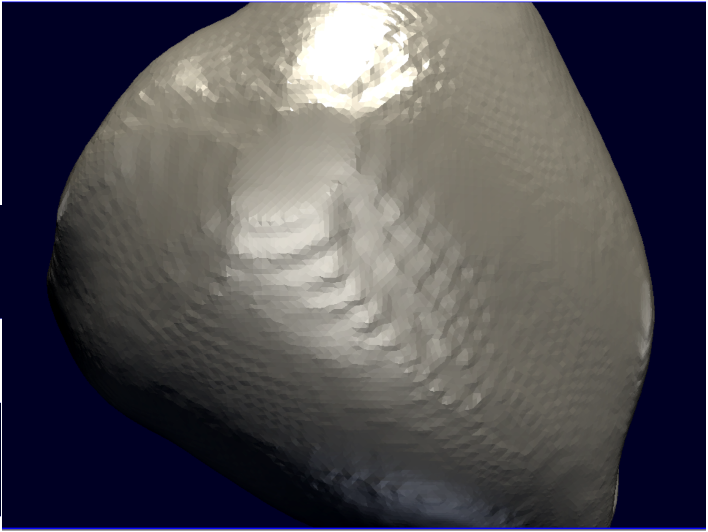

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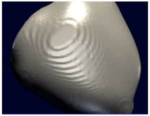

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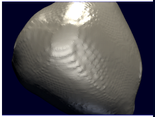

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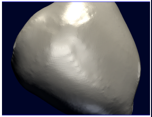

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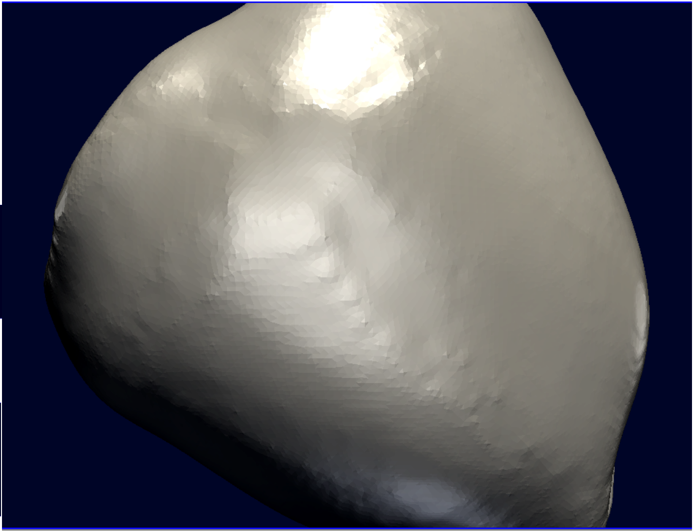

Overlay:

Quantitative results

Example 2: Outer iterations 2, inner iterations 200, $\alpha = 1/3$.

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Skewness P: Volume ratio to circumscribed ball.

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Percentiles of P	1%	5%	10%
Original mesh	0.0900	0.2350	0.3348
First iteration	0.1136	0.2360	0.3195
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Determinant ratio Θ: Measures change of size and orientation.

Percentiles of Θ	1%	5%
1. Iteration	0.6486	0.7407
2. Iteration	0.5280	0.6470

Outlook

- Update of interior points.

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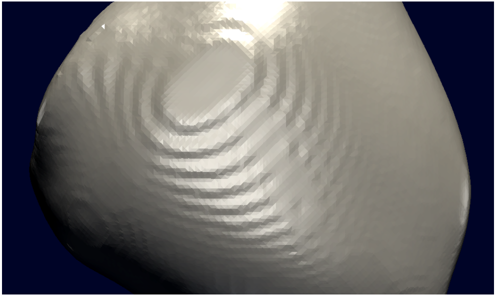

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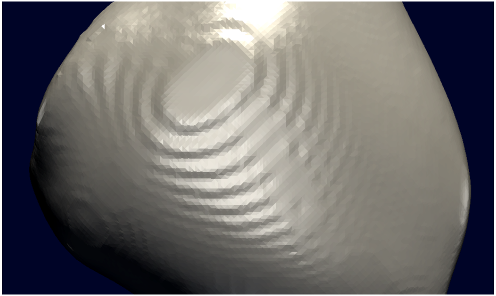

- More general constraints.

Outlook

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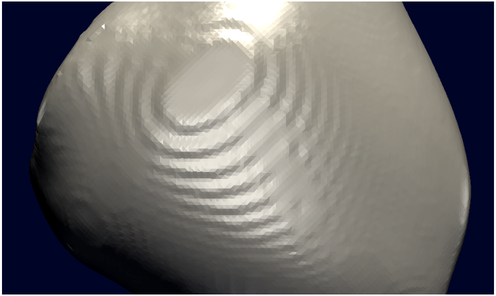

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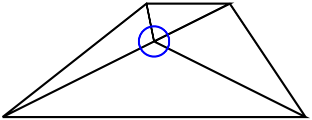

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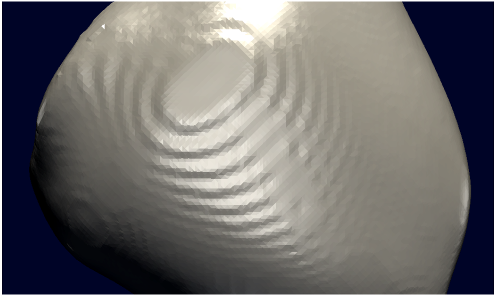

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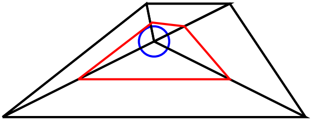

Thank you for your attention!

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